

# COSMOLOGICAL DYNAMICAL SYSTEMS

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# Contents

<b>Preface</b>	<b>xvii</b>
<b>Introduction</b>	<b>1</b>
<b>1 A bird's eye view on cosmology and cosmological problems</b>	<b>13</b>
1.1 Cosmological models . . . . .	13
1.1.1 Homogeneity and isotropy: the Roberson-Walker metric . . . . .	16
1.1.2 Dynamics: the Friedmann equations . . . . .	17
1.1.3 Flat universes . . . . .	20
1.1.4 Including curvature . . . . .	21
1.1.5 Geometry, destiny and dark energy . . . . .	22
1.1.6 Scalar fields and Dark Energy . . . . .	23
<b>2 Qualitative theory of dynamical systems</b>	<b>45</b>
2.1 Introduction . . . . .	45
2.2 Qualitative theory of dynamical systems . . . . .	48
2.2.1 Definitions and basic results . . . . .	48
2.2.2 Desirable stability properties of nonlinear vector fields . . . . .	51
2.2.3 Linearization . . . . .	53
2.2.3.1 Linear stability theorem . . . . .	55
2.2.4 Flow for autonomous vector fields . . . . .	57
2.2.5 Invariant sets . . . . .	59
2.2.5.1 Stable, Unstable and Center subspaces of singular points of linear autonomous vector fields . . . . .	60
2.2.5.2 Stable, Unstable and Center manifolds of singular points of nonlinear autonomous vector fields . . . . .	64
2.2.5.3 Center Manifold Theory . . . . .	66
2.2.5.4 Normal Forms . . . . .	68
2.2.5.5 Asymptotic behavior . . . . .	71
2.3 Procedure for analyzing cosmological dynamical systems . . . . .	76

<b>3</b>	<b>Non-minimally Coupled Dark Energy Models</b>	<b>79</b>
3.1	Introduction . . . . .	79
3.2	The Field Equations . . . . .	81
3.3	Late Time Behavior . . . . .	84
3.4	Early time behavior in the invariant set $\rho_r = 0$ . . . . .	92
3.4.1	The Topological Properties of the Phase Space . . . . .	94
3.4.2	The flow on the invariant set $(\partial\Sigma)_1$ . . . . .	98
3.4.2.1	Powerlaw-coupling function . . . . .	99
3.4.3	The flow on the invariant set $(\partial\Sigma)_2$ . . . . .	100
3.4.3.1	The Albrecht-Skordis potential . . . . .	102
3.4.4	The flow in the invariant set $\rho_r = 0$ as $\phi \rightarrow +\infty$ . . . . .	105
3.4.4.1	Location, existence and stability conditions of the singular points. Cosmological parameters . . . . .	108
3.4.5	The flow in the invariant set $\rho_r = 0$ near $\phi = -\infty$ . . . . .	112
3.4.6	The topological structure of the invariant set $\rho_r = 0$ at the past attractor . . . . .	113
3.4.6.1	The initial space-time singularity. . . . .	114
3.4.6.2	A global singularity theorem . . . . .	124
3.5	Early-time behavior for the model including radiation . . . . .	125
3.5.1	Normalized Variables and Dynamical System . . . . .	125
3.5.2	The Topological Properties of the Phase Space . . . . .	126
3.5.3	Monotonic Functions . . . . .	129
3.5.4	Singular points with $\phi$ Bounded . . . . .	130
3.5.5	Center manifold of $Q_2$ . . . . .	133
3.5.6	Analysis in the Limit $\phi \rightarrow \infty$ . . . . .	143
3.5.6.1	Singular points of the flow of (3.171)-(3.174) in the phase space (3.175). . . . .	144
3.5.6.2	Physical description of the solutions and connection with observables. . . . .	149
3.5.7	The Flow as $\phi \rightarrow -\infty$ . . . . .	151
3.6	Examples . . . . .	152
3.6.1	Numerical Evidence of the Result of Theorem 26 . . . . .	152
3.6.2	Coupling Functions and Potentials of Exponential Orders $M = 0$ and $N = -\mu \neq 0$ , Respectively . . . . .	152
3.6.2.1	Powerlaw coupling and Albrecht-Skordis potential in the invariant set $\rho_r = 0$ . . . . .	154
3.6.2.2	Powerlaw coupling and Albrecht-Skordis potential for the general model including radiation . . . . .	158
3.6.3	Quadratic Gravity: $F(R) = R + \alpha R^2$ . . . . .	159

3.6.3.1	Stability Analysis of the <i>de Sitter</i> Solution in Quadratic Gravity . . . . .	163
3.6.4	$R^n$ -Gravity . . . . .	169
3.7	Conclusion . . . . .	173
<b>4</b>	<b>Phantom dark energy with varying-mass dark matter particles</b>	<b>179</b>
4.1	Introduction . . . . .	179
4.2	Phase-space analysis . . . . .	181
4.2.1	Model 1: Exponential potential and exponentially-dependent dark-matter particle mass . . . . .	182
4.2.1.1	Finite analysis . . . . .	182
4.2.1.2	Analysis at infinity . . . . .	184
4.2.1.3	Cosmological implications and discussion: Model 1 . .	187
4.2.2	Model 2: Power-law potential and power-law-dependent dark-matter particle mass . . . . .	189
4.2.2.1	Finite analysis . . . . .	190
4.2.2.2	Stability of de Sitter solution for Power-law potential and power-law-dependent dark-matter particle mass. . .	190
4.2.2.3	Analysis at infinity . . . . .	194
4.2.2.4	Cosmological implications and discussion: Model 2 . .	201
4.2.3	Model 3: Power-law potential and exponentially-dependent dark-matter particle mass . . . . .	203
4.2.3.1	Finite analysis . . . . .	203
4.2.3.2	Stability of de Sitter solution for Power-law potential and exponentially-dependent dark-matter particle mass. .	205
4.2.3.3	Analysis at infinity . . . . .	206
4.2.3.4	Cosmological implications and discussion: Model 3 . .	212
4.2.4	Model 4: Exponential potential and power-law-dependent dark-matter particle mass . . . . .	213
4.2.4.1	Finite analysis . . . . .	213
4.2.4.2	Stability analysis of the phantom dominated solution for the exponential potential and power-law-dependent dark-matter particle mass . . . . .	214
4.2.4.3	Analysis at infinity . . . . .	217
4.2.4.4	Cosmological implications and discussion: Model 4 . .	225
4.3	Conclusions . . . . .	226
<b>5</b>	<b>Phase-space analysis of Hořava-Lifshitz cosmology</b>	<b>229</b>
5.1	Introduction . . . . .	229

5.1.1	Detailed Balance . . . . .	230
5.1.2	Beyond Detailed Balance . . . . .	232
5.2	The cosmological equations . . . . .	234
5.3	Detailed balance: Phase-space analysis . . . . .	236
5.3.1	Case 1: Flat universe with $\Lambda = 0$ . . . . .	237
5.3.1.1	Finite analysis . . . . .	238
5.3.1.2	Analysis at infinity . . . . .	239
5.3.1.3	Cosmological implications for Case 1: flat universe with $\Lambda = 0$ . . . . .	242
5.3.2	Case 2: non-flat universe with $\Lambda = 0$ . . . . .	243
5.3.2.1	Finite analysis . . . . .	243
5.3.2.2	Analysis at infinity . . . . .	244
5.3.2.3	Cosmological implications for Case 2: non-flat uni- verse with $\Lambda = 0$ . . . . .	246
5.3.3	Case 3: flat universe with $\Lambda \neq 0$ . . . . .	248
5.3.3.1	Finite analysis . . . . .	248
5.3.3.2	Analysis at infinity . . . . .	250
5.3.3.3	Cosmological implications for Case 3: flat universe with $\Lambda \neq 0$ . . . . .	252
5.3.4	Case 4: $k \neq 0, \Lambda \neq 0$ . . . . .	253
5.3.4.1	Finite analysis . . . . .	254
5.3.4.2	Analysis at infinity . . . . .	254
5.3.4.3	Cosmological implications for Case 4: non-flat uni- verse with $\Lambda \neq 0$ . . . . .	255
5.4	Beyond detailed balance: phase space analysis . . . . .	256
5.4.1	Stability Analysis of the <i>de Sitter</i> Solution in Hořava-Lifshitz cos- mology . . . . .	258
5.4.2	Cosmological implications: Beyond detailed balance . . . . .	260
5.5	Conclusions . . . . .	261
<b>6</b>	<b>Cardassian Cosmologies</b>	<b>263</b>
6.1	Introduction . . . . .	263
6.2	Field equations . . . . .	264
6.3	Phase-space analysis . . . . .	265
6.4	Basic observables . . . . .	269
6.5	Physical interpretation . . . . .	270
6.6	Cosmological consequences . . . . .	270
6.7	Conclusions . . . . .	271





# List of Figures

- 1.1 Two-dimensional contour-plots of the DE equation-of-state parameters, in two different parameterizations and using SNIa data. The left graph corresponds to ansatz A (expression (1.101)) and the right graph are to ansatz B (expression (1.102)). From Ref. [165]. . . . . 41
- 1.2 Two-dimensional contour-plot of the DE equation-of-state parameters, in parameterization ansatz B (expression (1.102)), and using WMAP, BAO, SNIa data. From Ref. [323]. . . . . 42
- 3.1 Projection of orbits on the phase plane  $(x_1, \varphi)$  for the coupling function (3.61): (a) for  $n = 2$ ,  $\gamma = 1.35$  and  $\chi_0 = 0.05$  the origin is an stable focus. (b) for  $n = 2$ ,  $\gamma = 1.4$  and  $\chi_0 = 3$  the origin is an stable node. (c) for  $n = 2$ ,  $\gamma = 1$  and  $\chi_0 = 0.3$  the origin is saddle point. Notice that the singular points  $(x_1, \varphi) = (0, \pm 1)$  seems to be saddle points for the cases (a) and (b) whereas in the case (c) they are local sinks. Observe that the singular points  $(x_1, \varphi) = (1, -1)$  and  $(y, \varphi) = (-1, 1)$  are in all the cases local sources (which is a suggestive argument in favor for the unboundedness of the scalar field into the past). . . . . 99
- 3.2 Phase plane  $(x_1, \varphi)$  for the model with potential (3.66) (a) for  $A = 3.25$ ,  $B = -2$ , and  $\mu = 0.5$  the singular points of the system are a saddle located at  $(x_1, \varphi) = (0, 0.6993)$  and a node at  $(x_1, \varphi) = (0, -0.6993)$ . (b) for  $A = 0.62$ ,  $B = 0.79$ , and  $\mu = -1.12$  the singular points of the system are a saddle located at  $(x_1, \varphi) = (0, -0.4806)$  and an stable spiral at  $(x_1, \varphi) = (0, 0.3078)$ . . . . . 101
- 3.3 Qualitative dynamics of the flow of (3.162) for the choice  $M = \sqrt{2/3}$  and  $\gamma = 1$ . Observe that  $P_1^+$  or  $P_1^-$  are the local attractors (sinks).  $P_3^+$  is a local source,  $P_3^-$  is a saddle for the full dynamics, but it is a local source in the invariant set  $\tanh \sigma_1 = -1$ . The thick dashed line is an invariant set which is unstable. In fact all its points including  $R_1^+$  and  $R_1^-$ , act as saddle points. They correspond to cosmological radiation-dominated solutions. . 153



3.4	The graphic illustrates the result of theorem 26. We set $M = \sqrt{2/3}$ and $\gamma = 1$ . The point with $\tanh \sigma_1 = \pm 1$ are the local sinks (thus for the original system (3.134)-(3.138) the scalar field almost always diverges towards the past). . . . .	154
3.5	Orbits in the invariant set $\{x_2 = 0\} \subset \bar{\Sigma}_\epsilon$ for the coupling function (3.183) and potential (3.66). . . . .	156
3.6	Orbits in the invariant set $\{\varphi = 0\} \subset \bar{\Sigma}_\epsilon$ for the coupling function (3.183) and potential (3.66). . . . .	157
3.7	Some orbits in the invariant set $\sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1$ for the choice of $\varphi = 0$ for the model with coupling function (3.183) potential (3.66). We select the values of the parameters: $\gamma = 1, \epsilon = 1.00, \mu = 2.00, A = 0.50, \alpha = 0.33, B = 0.5$ , and $\phi_0 = 0$ . . . . .	160
3.8	Some orbits in the invariant set $\sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1$ for the choice of $\varphi = 0$ for the model with coupling function (3.183) potential (3.66). We select the values of the parameters: $\gamma = 1, \epsilon = 1.00, \mu = 2.10, A = 0.50, \alpha = 0.33, B = 0.5$ , and $\phi_0 = 0$ . . . . .	161
3.9	Projection of some orbits of (3.171)-(3.174) in the invariant set $\varphi = 0$ for the coupling function (3.191) and the potential (3.190) for $\gamma = 1$ . Observe that $P_1$ and $P_2$ are local sources, $R_1$ , and $P_3$ are saddles ( $P_3$ is the local attractor in the invariant set $y = 0$ ) and $P_4$ (the de Sitter solution) is the local sink in the invariant set $\varphi = 0$ . . . . .	162
3.10	Projection of some orbits of (3.199) in the space $y_1, y_2, y_3$ for the coupling function (3.191) and the potential (3.190) for $\gamma = 1$ . The graphic shows the behavior in the stable manifold of $P_4$ . The bulk of orbits in front of and at the right hand side of the figure represents a projection of the center(s) manifold(s). . . . .	165
3.11	Projection of some orbits of (3.199) in the space $x, y_1, y_3$ for the coupling function (3.191) and the potential (3.190) for $\gamma = 1$ . The graphic shows the unstable character of $P_4$ (trajectories starting at $x < 0$ move away from the origin). . . . .	166
3.12	Projection of some orbits of (3.199) in the space $x, y_2, y_3$ for the coupling function (3.191) and the potential (3.190) for $\gamma = 1$ . The graphic shows the unstable character of $P_4$ (the orbits depart from the origin for $x < 0$ ). .	167

3.13	Projection in $\varphi = 0$ of some orbits of the flow of (3.171)-(3.174) for $M = \sqrt{2/3}$ , $N = -\frac{\sqrt{\frac{2}{3}(n-2)}}{n-1}$ . We set $n = 1.251$ . Observe that $R_1$ , $R_3$ are in the region of physical interest. These are saddle points. The singular points $P_4$ and $P_3$ exist and are saddle points. $P_1$ and $P_2$ are local sources and $P_5$ is a local sink. We display some orbits in the halfspace $\sigma_4 < 0$ (corresponding to contracting universes) for aesthetical purposes. Observe that $P_6$ mirrors the behavior of $P_5$ . . . . .	172
3.14	Projection of the orbits displayed in figure 3.13 to $\sigma_5 = 0$ . . . . .	173
4.1	Phase plane of Model 1 for the parameter values $\lambda_1 = 0.4$ and $\mu_1 = 2$ . The stable manifold of B (thick curve) divides the physical part of the phase space (region corresponding to $0 \leq \Omega_\phi \leq 1$ , bounded by the dashed (red) curves) in two regions. The orbits initially below this curve converge towards C. The orbits initially above this curve converge to A. [Taken from [152]; published with permission of Elsevier B.V]. . . . .	184
4.2	Phase plane of Model 1 for the parameter values $\lambda_1 = 1$ and $\mu_1 = 0.5$ . In this case the critical point B does not exist and all orbits initially at the physical region converge to A. The dashed (red) curves bound the physical part of the phase space, that is corresponding to $0 \leq \Omega_\phi \leq 1$ . [Taken from [152]; published with permission of Elsevier B.V]. . . . .	185
4.3	Poincaré (global) phase plane of Model 1 for the parameter values $\lambda_1 = 0.4$ and $\mu_1 = 2$ . The attractors in the finite region are A which is physical ( $0 \leq \Omega_\phi \leq 1$ ) and C. The orbits initially above the stable manifold of B converge to A (see figure 4.1). The points at infinity $Q_1$ and $Q_4$ are sources, whereas $Q_2$ and $Q_3$ are saddles. . . . .	187
4.4	Poincaré (global) phase plane of Model 1 for the parameter values $\lambda_1 = 1.0$ and $\mu_1 = 0.5$ . The attractor in the finite region is the singular point A which is physical ( $0 \leq \Omega_\phi \leq 1$ ) (see figure 4.2). The unphysical state C is a saddle. The points at infinity $Q_2$ and $Q_3$ are sources, whereas $Q_1$ and $Q_4$ are saddles. . . . .	188
4.5	xy-projection of the phase-space of Model 2, for the parameter values $\lambda_2 = -0.5$ and $\mu_2 = 0.5$ . The critical point E (representing de Sitter solutions) is the attractor of the system. The dashed (red) curves bound the physical part of the phase space, that is corresponding to $0 \leq \Omega_\phi \leq 1$ . [Taken from [152]; published with permission of Elsevier B.V]. . . . .	194
4.6	Poincaré (global) phase-space of Model 2, for the parameter values $\lambda_2 = -0.5$ and $\mu_2 = 0.5$ . The critical point E (representing de Sitter solutions) is a local attractor for the points at the finite region. The points at infinity $Q_{6,7,8,9}$ are local sources; $Q_{10,11,12}$ are saddles; and $Q_{13}$ is a local sink. . .	202

4.7	Poincaré (global) phase-space of Model 2, for the parameter values $\lambda_2 = 2.01 + \sqrt{30}$ and $\mu_2 = 0.5$ . The critical point E (representing de Sitter solutions) is a saddle point. The points at infinity $Q_{6,7,8,9,10}$ are local sources; $Q_{12,13}$ are saddles and $Q_{11}$ is a sink. . . . .	203
4.8	xy-projection of the phase-space of Model 3 for the parameter values $\lambda_2 = 1$ and $\mu_1 = 1.8$ . The stable manifold of G (thick curve) divides the physical part of the phase space (region bounded by the dashed (red) curves) in two regions. The orbits initially below this curve converge towards H, while those initially above this curve converge towards F. [Taken from [152]; published with permission of Elsevier B.V.]. . . . .	204
4.9	Poincaré (global) phase-space of Model 3, for the parameter values $\lambda_2 = -0.01$ and $\mu_1 = 1.8$ . For this choice of parameters $Q_{15}$ and $Q_{16}$ are local sources; $Q_{14}$ is unstable (of saddle type); $Q_{17,18,19,20}$ are saddles in the infinite region; $G, H$ are saddles in the finite region; $Q_{21}$ is a sink in the infinite region and $F$ is locally asymptotically stable. . . . .	212
4.10	xy-projection of the phase-space of Model 4 for the parameter values $\lambda_1 = 1$ and $\mu_2 = 1.8$ . The critical point J (corresponding to a super-accelerating universe) attracts all the orbits in this invariant set. The dashed (red) curves bound the physical part of the phase space, that is corresponding to $0 \leq \Omega_\phi \leq 1$ . [Taken from [152]; published with permission of Elsevier B.V.]. . . . .	214
4.11	Poincaré (global) phase-space of Model 4, for the parameter values $\lambda_1 = 1.0$ and $\mu_2 = 2.1 + \sqrt{15}$ . In the figure the attractor in the finite region is $J$ . The points at infinity $Q_{25,26}$ are the local sources whereas $Q_{27,28,29}$ are saddles. . . . .	224
4.12	Projection of 4.11 on the plane $x_r$ - $y_r$ . . . . .	225
5.1	Phase plane for a flat universe with $\Lambda = 0$ (case 1), for the choice $s = 0.6$ . The singular points $P_1$ and $P_2$ are unstable (sources), while $P_3$ is a global attractor. [Taken from [151] and published with permission of IOP Publishing Ltd]. See the global phase space in figure 5.2. . . . .	240
5.2	Poincaré projection (global phase space) of the system (5.43)-(5.44) (case 1) for the choice $s = 0.6$ . Observe that the points at infinity $Q_{1,2,3}$ are saddles, whereas the finite point $P_3$ is the global attractor. . . . .	241
5.3	Global phase space of the system (5.43)-(5.44) (case 1) for the choice $s = -0.6$ . The points at infinity $Q_2$ and $Q_3$ are saddles. The point $Q_1$ is a local attractor at infinity (but it is unphysical) and $P_3$ is a local attractor at the finite region. . . . .	242

- 5.4 Phase plane for a non-flat universe with  $\Lambda = 0$  (case 2), for the choice  $s = \sqrt{3}$ . In this specific scenario the singular points  $P_3$  and  $P_{5,6,7,8}$  do not exist, while  $P_1$  and  $P_2$  are unstable (source and saddle respectively). [Taken from [151] and published with permission of IOP Publishing Ltd]. See the global phase space in figure 5.6. . . . . 244
- 5.5 Phase plane for a non-flat universe with  $\Lambda = 0$  (case 2), for the choice  $s = 0.6$ . In this specific scenario the singular point  $P_3$  is a local attractor, while  $P_{1,2}$  are unstable (sources) and  $P_{5,6}$  are saddle ones. [Taken from [151] and published with permission of IOP Publishing Ltd]. See the global phase space in figure 5.7. . . . . 245
- 5.6 Global phase space of the system (5.53)-(5.54) (case 2) for the choice  $s = \sqrt{3}$ . The singular points  $P_3$  and  $P_{5,6,7,8}$  do not exist. At finite region we find only the source  $P_1$  and the saddle  $P_2$ , and we indeed observe that there is one orbit approaching  $P_2$  (the solution with  $z \equiv 0$ ). The future attractors of the system are  $Q_5$  and  $Q_8$  located at the infinite region. The points at infinity  $Q_{4,6,7}$  are saddles.  $Q_4$  is unphysical since  $x \notin \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ . 247
- 5.7 Global phase space of the system (5.53)-(5.54) (case 2) for the choice  $s = 0.6$ . The finite points  $P_1$  and  $P_2$  are local sources.  $P_3$  is a local sink at the finite region. Observe that the orbits spent a finite lapse of time near the saddle points  $P_5$  and  $P_6$  before reaching the local sinks at infinity  $Q_5$  and  $Q_8$  respectively. The points at infinity  $Q_{4,6,7}$  are saddles.  $Q_4$  is unphysical since  $x \notin \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ . . . . . 248
- 5.8 Phase plane for a flat universe with  $\Lambda \neq 0$  (case 3), for the choice  $s = \sqrt{3}$ . In this specific scenario the singular point  $P_{11}$  does not exists.  $P_{10}$  is unstable (source), while  $P_9$  is a saddle one. [Taken from [151] and published with permission of IOP Publishing Ltd]. See the global phase space in figure 5.10. . . . . 249
- 5.9 Phase plane for a flat universe with  $\Lambda \neq 0$  (case 3), for the choice  $s = 0.6$ . In this specific scenario the singular point  $P_{11}$  is a saddle one, while  $P_9$  and  $P_{10}$  are unstable (sources). [Taken from [151] and published with permission of IOP Publishing Ltd]. See the global phase space in figure 5.11. . . . . 250
- 5.10 Global phase space of the system (5.43)-(5.44) (case 3) for the choice  $s = \sqrt{3}$ . As in figure 5.8, the singular point  $P_{11}$  does not exists.  $P_{10}$  is unstable (source), while  $P_9$  is a saddle one. The orbits passing near the saddle  $P_9$  bifurcates and tends asymptotically to one of the global attractor at infinity  $Q_{10}$  or  $Q_{13}$  depending on the sign of the initial value of  $u_r$ . The points at infinity  $Q_{9,11,12}$  are saddles;  $Q_9$  is unphysical. . . . . 252

5.11	Global phase space of the system (5.62)-(5.63) (case 3) for the choice $s = 0.6$ . As in figure 5.9, in this specific scenario the singular point $P_{11}$ is a saddle one, while $P_9$ and $P_{10}$ are unstable (sources). The global attractors at infinity are the points $Q_{10}$ and $Q_{13}$ . The points at infinity $Q_{9,11,12}$ are saddles; $Q_9$ is unphysical. . . . .	253
6.1	Phase space of Cardassian models for the choices: (a) $\gamma = 1.0, s = \sqrt{3} + 0.0001, n = 0.5$ . and (b) $\gamma = 1.0, s = \sqrt{3} + 0.0001, n = -0.5$ . . .	272
6.2	Phase space of Cardassian models for the choices: (a) $\gamma = 1.0, s = \sqrt{3} + 0.0001, n = 0.95$ . and (b) $\gamma = 1.0, s = \sqrt{3} + 0.0001, n = -0.95$ . .	273

# List of Tables

1.1	A summary of the behaviors of the most important sources of energy density in cosmology. . . . .	21
1.2	DE models based on scalar fields ( $X \equiv -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$ ). . . . .	25
3.1	Simple examples of WBI behavior at large $\phi$ . $n$ and $\lambda$ are arbitrary constants. Adapted from [403]. . . . .	107
3.2	The properties of the singular points for the system (3.78)-(3.80). We use the notations $\alpha = 3(N(\gamma - 2) + M(3\gamma - 4))$ , $\beta = 2(2N - M(3\gamma - 4))$ , $\delta = \frac{M(3\gamma - 4)}{\sqrt{6}(\gamma - 2)}$ , and $\Gamma = \frac{\sqrt{2(\gamma-2)(3\gamma-2)}}{4-3\gamma}$ . . . . .	109
3.3	Monotonic functions for the flow of (3.134)-(3.138) in (3.139). . . . .	129
3.4	Location of the singular points of the flow of (3.144)-(3.147) defined in $\Omega$ . . . . .	131
3.5	Observable cosmological quantities, and physical behavior of the solutions, at the singular points of the cosmological system. We use the notations $M_1(\gamma) = \frac{\sqrt{2\gamma(3\gamma-8)+8}}{4-3\gamma}$ , $M_2(\gamma) = \frac{\sqrt{6}\sqrt{(\gamma-3)\gamma+2}}{4-3\gamma}$ . . . . .	150
3.6	Location of the singular points of the flow of (3.171)-(3.174) defined in the invariant set $\{p \in \Omega_\epsilon : \varphi = 0\}$ for $M = 0$ and $N = -\mu$ . . . . .	155
3.7	Location of the singular points of the flow of (3.171)-(3.174) defined in the invariant set $\{p \in \Omega_\epsilon : \varphi = 0\}$ for $M = \sqrt{2/3}$ and $N = 0$ . . . . .	159
3.8	Location of the singular points of the flow of (3.171)-(3.174) defined in the invariant set $\{p \in \Omega_\epsilon : \varphi = 0\}$ for $M = \sqrt{2/3}$ and $N = -\frac{\sqrt{\frac{2}{3}(n-2)}}{n-1}$ . We use the notations $n_+ = \frac{1}{5}(4 + \sqrt{6})$ , $N_+ = \frac{2}{27}(11 + 2\sqrt{10})$ , $\Gamma(n) = -\frac{2n(n-2)}{3-9n+6n^2}$ , and $\Gamma_+(\gamma) = \frac{9\gamma+\sqrt{9\gamma^2+48\gamma+16}+4}{12\gamma+4}$ . . . . .	170
3.9	Stability of the singular points of the flow of (3.171)-(3.174) defined in the invariant set $\{p \in \Omega_\epsilon : \varphi = 0\}$ for $M = \sqrt{2/3}$ and $N = -\frac{\sqrt{\frac{2}{3}(n-2)}}{n-1}$ . We use the notations $n_+ = \frac{1}{5}(4 + \sqrt{6})$ , $N_+ = \frac{2}{27}(11 + 2\sqrt{10})$ , $\Gamma(n) = -\frac{2n(n-2)}{3-9n+6n^2}$ , and $\Gamma_+(\gamma) = \frac{9\gamma+\sqrt{9\gamma^2+48\gamma+16}+4}{12\gamma+4}$ . . . . .	171
4.1	The real and physically meaningful critical points of Model 1 and their behavior. . . . .	183
4.2	Basic observables and conditions for acceleration for the real and physically meaningful critical points of Model 1. . . . .	183

4.3	Asymptotic singular points of the system (4.10) (case 1) and their stability.	186
4.4	The real and physically meaningful critical points of Model 2 and their behavior. . . . .	190
4.5	Asymptotic singular points of the system (4.19) (case 2) and their stability. We use the notations $\alpha = \frac{3\sqrt{2}}{\sqrt{22-4\lambda_2+\lambda_2^2}}$ and $\beta = \frac{3}{\sqrt{13-4\mu_2+\mu_2^2}}$ , $\epsilon = \text{sign}(\lambda_2 - \mu_2)$ , $\delta = \text{sign}(-26 + 4\lambda_2 + \lambda_2^2)$ , $\eta = \text{sign}(-11 - 4\mu_2 + \mu_2^2)$ , and $\mu_- = \frac{1}{3} \left( 5 - \sqrt[3]{\frac{2}{245-9\sqrt{741}}} - \sqrt[3]{\frac{1}{2}(245 - 9\sqrt{741})} \right) \approx -0.47$ . NH stands for nonhyperbolic. . . . .	196
4.6	Basic observables for the singular points of the system (4.19) (case 2). Solution types . . . . .	201
4.7	The real and physically meaningful critical points of Model 3 and their behavior. . . . .	204
4.8	Asymptotic singular points of the system (4.49) (case 3) and their stability. We use the notations $\alpha = \frac{3\sqrt{2}}{\sqrt{22-4\lambda_2+\lambda_2^2}}$ , $\epsilon = \text{sign}(\lambda_2)$ , $\delta = \text{sign}(-26 + 4\lambda_2 + \lambda_2^2)$ . NH stands for nonhyperbolic. . . . .	208
4.9	Basic observables for the singular points of the system (4.49) (case 3). Solution types . . . . .	209
4.10	The real and physically meaningful critical points of Model 4 and their behavior. . . . .	214
4.11	Asymptotic singular points of the system (4.67) (case 4) and their stability. We use the notations $\beta = \frac{3}{\sqrt{13-4\mu_2+\mu_2^2}}$ , $\epsilon = \text{sign}(\mu_2)$ and $\eta = \text{sign}(-11 - 4\mu_2 + \mu_2^2)$ . NH stands for nonhyperbolic. . . . .	219
4.12	Basic observables for the singular points of the system (4.67) (case 4). Solution types. We use the notations $M(\mu_2) = -\frac{1}{9}(\mu_2 - 2)^2$ and $N(\mu_2) = \frac{1}{6}(-\mu_2^2 + 4\mu_2 - 1)$ . . . . .	220
5.1	Finite singular points of a flat universe with $\Lambda = 0$ (case 1) and their behavior. NH stands for nonhyperbolic (adapted from [151]). . . . .	239
5.2	Finite singular points and their corresponding solutions for a flat universe with $\Lambda = 0$ (case 1). . . . .	239
5.3	Asymptotic singular points of the system (5.43)-(5.44) (case 1) and their stability. . . . .	241
5.4	The singular points of a non-flat universe with $\Lambda = 0$ (case 2) and their behavior (adapted from [151]). . . . .	244
5.5	Asymptotic singular points of the system (5.53)-(5.54) (case 2) and their stability. . . . .	246
5.6	The singular points of a flat universe with $\Lambda \neq 0$ (case 3) and their behavior. NH stands for nonhyperbolic (adapted from [151]). . . . .	249

5.7	Asymptotic singular points of the system (5.62)-(5.63) (case 3) and their stability. . . . .	250
5.8	The singular points of a non-flat universe with $\Lambda \neq 0$ (case 4) and their behavior (adapted from [151]). . . . .	254
5.9	Asymptotic singular points of the system (5.72)-(5.72) (case 4) and their stability. NH stands for nonhyperbolic . . . . .	255
5.10	The singular points of a universe governed by Hořava gravity beyond detailed balance (system 5.82)) and their behavior. NH stands for nonhyperbolic (adapted from [151]). . . . .	257
5.11	Observable cosmological quantities of a universe governed by Hořava gravity beyond detailed balance. . . . .	258
6.1	Location and existence conditions, eigenvalues and dynamical character of the critical points of the dynamical system (6.8)-(6.10). We assume $n < 1$ , $0 < \gamma < 2$ and $s^2 < 6$ . We use the notation $\beta_{\pm} = -\frac{3}{4} \left( (2 - \gamma) \pm \sqrt{(2 - \gamma)(24\gamma^2/s^2 + (2 - 9\gamma))} \right)$ . . . . .	267
6.2	Deceleration parameter, $q$ , and effective EoS, $w_{\text{eff}}$ at the critical points of the dynamical system (6.8)-(6.10). . . . .	269





# Preface

Over the last century, the investigations in Gravitation and Cosmology have attracted the attention of thousands of human minds. Apart this boom in popularity, the homework task for cosmologists is to try to give an answer to the reasons why the present Universe has the properties that are observed. In spite of having a well-established theoretical framework like the Einstein's gravity theory (GR), which is valid in an amplest rank of energies and that has responded satisfactorily to numerous empirical verifications, are numerous unanswered intriguing questions. Is the dark energy the cause of the acceleration of the expansion? What is its equation of state nowadays? Why the energy densities of matter and dark energy are of the same order of magnitude nowadays? Will be the acceleration of the expansion an indication that the action of Einstein-Hilbert must be modified? Can be reached the isotropic degree of the universe that is observed today, independently of the initial degree of anisotropy? In this book are studied, from the perspective of the dynamical systems, several Universe models that try to give answers to some of these questions.

In chapter 1 we give a bird's eye view on cosmology and cosmological problems. Cosmology is a particular way to embrace the space-time, the gravity theory and the material content of the Universe, in order to clarify the origins and the evolution of the Universe as a whole. Within the most commonly accepted vision in Cosmology, it is assumed that the geometry of the universe at the large scale is described by General Relativity; although in this book we are ascribed also to alternative theories. In this chapter we put in a firmly theoretical setting the above questions.

Chapter 2 is devoted to a brief review on some results and useful tools from the qualitative theory of dynamical systems. They provide the theoretical basis for the qualitative study of concrete cosmological models. This is the more mathematical in character. However, to avoid a lengthy and tedious reading, we omit several of the mathematical proofs. By this reason, it is required that the reader will be managed with concepts from Non-linear Differential Equations, particularly, for the stability theory of singular points and periodic orbits, stable and unstable manifolds; basic rudiments on Differential Geometry, and a firm knowledge on Calculus. These skills, and a deeply understanding of Chapter 1, will be required to read the book to the end.

Chapters 1 and 2 are a review of well-known results. Chapters 3, 4, 5 and 6 are

devoted to our main results. In these chapters are extended and settled in a substantially different, more strict mathematical language, several results obtained by one of us in [136, 137, 152, 151] (at these stage we acknowledged to all the collaborators and to the Editorial houses; that gave their permissions for using several of these previous results). In chapter 6, we provide a different approach to the subject discussed in [452].

The aim of the chapter 3 is to extent several results related to flat FRW models formulated in the conformal (Einstein) frame of scalar-tensor gravitational theories, including  $F(R)$  theories through conformal transformation. We will focus mainly in a particular era of the universe where matter and radiation coexisted. The ordinary matter is described by a perfect fluid. We consider models with and without radiation. We are interested in investigating all possible scaling solutions in this regime, since scaling late-time attractors provide a hint for avoiding or alleviating the Coincidence Problem. Although we are mainly interested in describing the early time dynamics, for completeness we will focus also in the late-time dynamics of the models under consideration. For flat FRW models we obtain sufficient conditions under the potential for the asymptotic stability of the non-negative local minima for the potential. Center manifold theory is employed to analyze the stability solutions associated to the local degenerated minimum and to the inflection points of the potential. We prove, for arbitrary potentials and arbitrary coupling functions of appropriate differentiable class, that the scalar field almost always diverges into the past generalizing previous results. It is designed a dynamical system well suited to studying the stability of the singular points in that limit. We obtain there: radiation-dominated cosmological solutions; power-law scalar-field dominated inflationary cosmological solutions; matter-kinetic-potential scaling solutions and radiation-kinetic-potential scaling solutions. It is discussed, by means of several worked examples, the link between our results and the results obtained for specific  $F(R)$  frameworks by using appropriated conformal transformations. We prove an stability theorem and two singularity theorems.

In chapter 4, we investigate several varying-mass dark-matter particle models in the framework of phantom cosmology. We examine whether there exist late-time cosmological solutions, corresponding to an accelerating universe and possessing dark energy and dark matter densities of the same order. Imposing exponential or power-law potentials and exponential or power-law mass dependence, we conclude that the coincidence problem cannot be solved or even alleviated. We improve our previous analysis by using the Center Manifold Theory to analyze the stability of the nonhyperbolic fixed points in the phase space. Basically, we use these cosmological models as examples of how to apply the Center Manifold Theory in cosmology. Also, in this chapter we perform a Poincaré compactification process allowing to construct a global phase space containing all the cosmological information in both finite and infinite regions. Are proved several stability theorems.

Chapter 5 is devoted to a detailed phase-space analysis of Hořava-Lifshitz cosmol-

ogy, with and without the detailed-balance condition. Under detailed-balance we find that the universe can reach a bouncing-oscillatory state at late times, in which dark-energy, behaving as a simple cosmological constant, is dominant. In the case where the detailed-balance condition is relaxed, we find that the universe reaches an eternally expanding, dark-energy-dominated solution, with the oscillatory state preserving also a small probability. To achieve the above results we use the Center Manifold Theory to analyze the stability of the nonhyperbolic fixed points in the phase space. Also, we perform a Poincaré compactification process allowing to construct a global phase space containing all the cosmological information in both finite and infinite regions. We prove a stability theorem.

In chapter 6, we investigate the asymptotic behavior of Cardassian cosmological models filled with a perfect fluid and a scalar field with an exponential potential. Cardassian cosmologies arise from modifications of the Friedmann equation, and among the different proposals within that framework we will choose those of the form  $3H^2 - \rho \propto \rho^n$  with  $n < 1$ . We construct a three dimensional dynamical system arising from the evolution equations. Using standard dynamical systems techniques we find the fixed points and characterize the solutions they represent. We pay especial attention to the properties inherent to the modifications and compare with the (standard) unmodified scenario. Among other interesting results, we find there are no late-time scaling attractors. We prove the stability (but not asymptotic stability) of a cosmological solution representing a regime where the Cardassian corrections dominates.

This book is intended to a wide audience, specially to cosmologist and mathematicians working on the field of Gravitation and Cosmology. Most parts of the text are at the level of a last year math undergraduate student. Several parts of the book were used to complement the course “Dynamical Systems” taught last Spring by one of us (GL) to the 3rd and 4th year of the Mathematics degree at Universidad Central de Las Villas (Santa Clara, Cuba).

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Genly Leon and Carlos R. Fadragas

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# Introduction

Cosmology is a broad and promising area of research in applied Mathematics and Physics. It is supported by reliable observed data available in modern Astrophysical literature. As a mathematical discipline, its methods come from two main sources: Differential Geometry and Differential Equations. At the same time Cosmology returns new problems not only to both fields, also to related Physics.

There exist three essential elements whenever cosmological modelling is concerned: the space-time, the gravity theory and the material content of the Universe. Cosmology is a particular way to embrace these basic elements for studying the origins and the evolution of the Universe as a whole.

There exists a basic hierarchy of space-times in descendant degree of symmetry and, therefore, in an increasing degree of generality: isotropic and homogeneous Friedmann-Lemaître-Robertson-Walker (FLRW) space-time, homogenous space-time (Bianchi and Kantowsky-Sachs), inhomogeneous space-time and generic space-time (i.e. without symmetries).

Some of the theories of gravity are: General Relativity (GR), higher order gravity (HOG) or extended gravity (EG), Scalar-tensor theories (STT), String theory (ST), and others.

The matter fields have played important roles at different epochs in the Universe evolution: vacuum, fluids (barionic/non-barionic matter, dark matter, radiation), scalar fields, n-form fields, etc.

There exist four standard ways of systematic investigation that can be used to examining cosmological models: Obtaining and analyzing exact solutions; Heuristic approximation methods; Numerical simulation, and Qualitative analysis. In the last case can be used three different approaches: (a) Piecewise approximation methods, (b) Hamiltonian methods, (c) Dynamical systems methods. In the approach iv (a), the evolution of the model universe is approximated through a sequence of epochs in which certain terms in the governing differential equations can be neglected, leading to a simpler system of equations. This heuristic approach is firmly based in the existence of heteroclinic sequences, which is a concept from iv (c). In the approach iv (b), the Einsteins equations are reduced to a Hamiltonian system dependent of time for a particle (point universe) in two dimensions. This approach has been used mainly for modelling and analyzing the dynamics of

the universe nearly the Big-Bang singularity. In the approach iv (c), Einstein's equations for homogeneous cosmologies can be described as an autonomous system of first order ordinary differential equations plus certain algebraic constraints. In this case the solution curves form a partition  $\mathbb{R}^n$  in orbits, defining a dynamical system in  $\mathbb{R}^n$ . The approach that is used is to start from local analysis and to be extending, step by step, the regions of the state space and the parameter space that are investigated. In the general case the sets of the partition of the state space can be enumerated and described. This study consists of several steps: the determination of the singular points, the linealization in a vicinity of them, the search of the eigenvalues of associated the Jacobian matrix, the verification of the conditions of stability in a vicinity of the singular points with physical sense, the finding of the their sets of stability and instability and the determination of their basin of attraction. Using this approach in [1] have been obtained many results concerning the possible asymptotic cosmological states in Bianchi and FRW models, whose material content is a perfect fluid (usually modelling "dark matter" (MO), a component that plays an important role in the formation of structures in the Universe, such as galaxies and clusters of galaxies) with linear equation of state (with the possible inclusion of a cosmological constant). Also several classes of inhomogeneous models are examined comparing the results with those obtained using numerical and Hamiltonian methods. This analysis is extended in [2], to other contexts, having considered other material sources such as the scalar fields.

The investigations in Gravitation and Cosmology try to give an answer to the reasons why the present Universe has the properties that are observed. In spite of having a well-established theoretical framework like the Einstein's gravity theory (GR), who is valid in an amplest rank of energies and that has responded satisfactorily to numerous empirical verifications, are numerous unanswered intriguing questions. Is the dark energy the cause of the acceleration of the expansion? What is its equation of state nowadays? Why the energy densities of matter and dark energy are of the same order of magnitude nowadays? Will be the acceleration of the expansion an indication that the action of Einstein-Hilbert must be modified? Can be reached the isotropic degree of the universe that is observed today, independently of the initial degree of anisotropy?

In order to respond them the consensus says that it is necessary to progress in the theoretical and/or phenomenological modelling of the Universe on the basis of an increasing number of observational data that inform to us into how it is the Universal kinematics on great scales, and on the other hand, in the deepening in the understanding of the fundamental theory describing the gravitational interaction.

In this proposal are studied, from the perspective of the dynamical systems, several models of the Universe as they try to give answer to some of these questions. These techniques allow the fine tuning of the initial conditions required to conciliate with the observations.

There exists a huge amount of astrophysical data, collected since 1998 to the date, that are the basis of a new cosmological paradigm, according to that, the universe is spatially flat and it seems to be in an accelerated expansion phase. Strong evidences for that coming from the Hubble diagram, the Supernovae type Ia observations of velocity-luminosity, and the anisotropies observed in the cosmic background radiation [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

It is convenient to measure the energy density of the different species in terms of the critical density  $\rho_c(t_0) = 3H_0^2/8\pi G$ , where  $H_0 = (\dot{a}/a)_0$  is the current expansion rate of the universe. The critical density is exactly the required density to make the universe flat. If the energy density in the universe is lower than the critical one, the universe will expand up to a maximum value and then it will recollapsing; if it is lower, the universe would expand forever. Following the recent measurements, the critical density is given by  $\rho_c(t_0) = 1.88h^2 \times 10^{-26} kg m^{-3}$ , where  $h = 0.72 \pm 0.007$  [20, 21]. The dimensionless energy density parameter  $\Omega_i = \rho_i/\rho_c$ , allows us to know the contribution of the different energetic components in the Universe to the total energy density (where  $i$  is used to denote the  $i$ -th component, e.g., matter (M), dark energy (DE), radiation (R), etc.).

- The total energy density is bounded by  $0.98 \lesssim \Omega_{total} \lesssim 1.08$ . This value is determined by the angular anisotropy spectrum in cosmic microwave background radiation (CMB). These observations, combined with the reasonable hypothesis  $h > 0.5$ , shows that we live in a universe with the critical density, that is a flat universe.
- The observations of primordial deuterium originated in the Big-Bang nucleosynthesis as well as the CMB observations shows that baryons contribute around  $\Omega_B = (0.024 \pm 0.0012)h^{-2}$ ; since  $h = 0.72 \pm 0.007$ ,  $\Omega_B \cong 0.04 - 0.06$ ; thus, we conclude that most of the matter content in the universe is non-baryonic.
- The observations related to large scale structures (LSS), as well as they dynamics (rotation curves of galaxies, the estimate of mass in clusters of galaxies, gravitational lensing, galactic surveys ...), suggest that our universe contains a non-luminous matter (dark matter; DM hereafter) composed by weakly interacting massive particles (WIMPs) which does cluster at galactic scales. This matter source contributes about  $\Omega_{DM} \cong 0.20 - 0.35$ .
- Combining the last observation with the first one, we conclude that there exists at least one additional component in the cosmic budget that contributes about 70% of the critical density. The initial analysis of some observations [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], indicates that this energy-density source (called dark energy; DE hereafter) is unclustered, has negative pressure and contributes to the total content as  $\Omega_{EO} \cong 0.60 - 0.75$ . The observations suggest that this component has the equation of state (EoS) parameter  $w = p/\rho \lesssim -0.78$

- The universe also contains radiation contributing an energy density  $\Omega_R h^2 = 2.56 \times 10^{-5}$  ( $\Omega_R \cong 4.84 - 5.03 \times 10^{-5}$ ). Today much of such radiation is due to the photons in the CMB radiation. Its contribution is dramatically irrelevant today but it would have been the dominant component in the universe for redshifts larger than  $z_e \cong \Omega_{MO}/\Omega_R \cong 4 \times 10^4 \Omega_{MO} h^2$ .
- Thus, we conclude that:

$$(\Omega_{EO}, \Omega_{MO}, \Omega_B, \Omega_R) \cong (0.7, 0.26, 0.04, 5 \times 10^{-5}).$$

In conclusion, we are facing the fact that the 96% of the total energetic content of our universe consist of energy/matter forms whose nature is intriguingly unknown; thus, time and resources are required to solve such an enigma. From this matter/energy, the 70% counteracts gravity and it is responsible for the accelerated expansion phase our universe is experiencing. In the vast literature on the subject, there are numerous descriptions which have more and less arguments in favor, without achieving a definitive answer.

These features of the universe led physicists to follow two directions in order to explain the accelerated expansion. The first is to introduce the concept of dark energy (see the reviews [23, 24, 25, 26, 138] and references therein) in the right-hand-side of the field equations, which could either be the simple cosmological constant  $\Lambda$ <sup>1</sup> or, one or several scalar fields [31, 32, 33, 34, 35]. The second is looking for alternative models [36, 37, 38].

Within the orthodox vision, the dynamics of the universe is well described by the Einstein's equations for the gravitational field. These relates the geometry of spacetime, to its matter content. The conventional treatment in cosmology separates the study of the universe in the large scales ( $l \gtrsim 150h^{-1}Mpc$ ) from the study of structure formation in lower scales. The first is modelled as an homogeneous and isotropic matter distribution (Cosmological Principle) and the second is solved in terms of gravitational instabilities which can amplify the small initial density perturbations, leading to the formation of structures like galaxies.

The Cosmological Principle allows for a substantial reduction of the complexity of Einstein's field equations; particularly it is reduced the number of independent equations. Exactly in such an approximation, the universe expansion is described by a function of time  $a(t)$  (called scale factor, whose time derivative will be denoted by  $\dot{a}(t)$ ) and its evolution is governed by the equations (we have used units in which  $c = 1$ ):

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G\rho}{3}; \quad d(\rho a^3) = -p da^3. \quad (1)$$

The first of them (called Friedmann equation) relates the expansion rate with the energy

---

<sup>1</sup>This choice is seriously plagued by the well known coincidence and fine tuning problems [27, 28, 29, 30].



density  $\rho$  ( $k = 0, \pm 1$  is the parameter characterizing the spatial curvature of the universe). The second one, determines the evolution of the density  $\rho = \rho(a)$  in terms of the scale factor if it is considered an EoS  $p = p(\rho)$ . Particularly, if  $p = w\rho$  with  $w$  (at least approximately) a constant, then  $\rho \propto a^{-3(1+w)}$  and, assuming  $k = 0$ ,  $a \propto t^{2/[3(1+w)]}$ .

Additionally, the spatial part  $\mathbf{g}$  of the geodesic acceleration (this one measures the relative acceleration between two geodesics in the spacetime) satisfies an exact equation in general relativity given by

$$\nabla \cdot \mathbf{g} = -4\pi G(\rho + 3p). \quad (2)$$

This shows that the source of the geodesic acceleration is  $(\rho + 3p)$  and not  $\rho$ . Thus, if  $\rho + 3p < 0$  matter exerts a negative pressure that counteracts the action of gravity driving the current accelerated expansion. This will be achieved if matter satisfies  $\rho > 0$  and  $w = p/\rho < -1/3$ . If we consider a sufficiently large scale such that the Cosmological Principle can be considered as valid, then, equation (2) reduces to

$$3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p). \quad (3)$$

As a consequence of the previous discussion, the acceleration (deceleration) of the universe can be characterized by the sign of  $\ddot{a}$ . If it is positive (negative) expansion is accelerated (decelerated).

To characterize the current expansion, it is used the “deceleration” factor measured today,  $q_0$ , which can be identified in the Taylor expansion:

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t - t_0) - \frac{q_0}{2}H_0(t - t_0)^2 + \dots \quad (4)$$

This defines

$$q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)} \frac{1}{H_0^2} = -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}(t_0)^2}, \quad (5)$$

where the overdot denotes derivative with respect time; the subindex 0 refers to magnitudes measured at current time ( $a(t_0)$  is the value of the scale factor today).

Knowing the properties of matter in the universe, then  $q_0$  is not independent with respect to the parameters  $H_0$  and  $\Omega_0$ . These two parameters are sufficient (at least in first approximation) to describe all the possibilities. However, if we do not know the properties of matter in the universe,  $q_0$  provides a complementary information.  $q_0$  can be measured directly by observing distant objects such as distant galaxies.

Perhaps the alternative to consider an scalar field to explain the current acceleration should be a more interesting possibility since this field appears in Grand Unification theories (GUT); and they are based in fundamental physics. However, they have its own

drawbacks [39]:

- This models are degenerated in several senses:
  - virtually any form of  $a(t)$  can be modelled by a suitable “designer”  $V(\phi)$ ;
  - even when  $w(a)$  is known, it is not possible to univocally determine the nature of the scalar field Lagrangian that gives it origin. At least two different forms of the scalar field Lagrangian (corresponding to quintessence or tachyonic field) could lead to the same  $w(a)$  (see the explicit construction in [39]).
- All the scalar field potentials require fine tuning of the parameters in order to be viable. This is obvious in the quintessence models in which adding a constant to the potential is the same as invoking a cosmological constant. So to make the quintessence models to work, we first need to assume the cosmological constant is zero.
- By and large, the potentials used in the literature have no natural field theoretical justification. All of them are non-renormalizable in the conventional sense and has to be interpreted as a low energy effective potential in an ad hoc manner.
- Although the large astrophysical observations are consistent with the  $\Lambda$ -CDM model ( $w = -1$ ) some of them favor the case  $w < -1$ . If that is true, the EoS parameter should crosses the phantom barrier ( $w = -1$ ).

Scalar fields have been also proposed to modelling dark matter in the halo of spiral galaxies, where the centers predicted by such models are scalar field condensates and its density profiles are almost constants [40]. The scalar field can gravitationally influence the galaxy causing that the orbital velocity of the celestial objects within this region remains a constant, explaining the flatness of rotational curves of galaxies. This fields can model also larger structures like clusters and super-clusters of galaxies. This fact somewhat justify that we consider an scalar field modelling dark matter in our Cardassian model.

In Cardassian cosmologies [41] is enough to consider a universe filled with cosmic dust (pressureless cold dark matter) and perhaps radiation, to modelling an expanding universe with geometric origin.

The dark energy can be described using scalar, vector or spinorial fields. Within the scalar matter, the quintessence field is the more investigated nowadays. Another possibility are the phantom fields. This models implies an “extraneous physics” because its kinetic energy is negative and there is also quantum instability. However, the hybrid model called quintom gives a dynamics for its equation of state favored by recent astrophysical observations [138, 42].

Since 2004 to the date, researchers have deserved several journal pages for the construction of dark energy models based in field theory which allows for the phantom-barrier

crossing. First than all, a single scalar field with canonical kinetic energy (quintessence) or non-conventional kinetic energy (phantom) do not gives the desired result [43, 44] (unless we consider non-minimally coupled scalar field models based in Scalar-tensor theories [45, 46, 47, 48, 49, 50]).

The crossing of the phantom divide is a significant challenge for theoretical physics. It was proved that the EoS of dark energy cannot cross the phantom divide if 1.) the dark energy component has an arbitrary scalar field Lagrangian, which has a general dependence on the field itself and its first derivatives., 2.) general relativity holds and 3.) the geometry of the universe is the spatially flat Friedmann-Robertson-Walker [44]. Thus, realizing such a crossing is not a trivial work.

To cross the phantom divide, we must break at least one of the conditions enumerated above. The more simply way to do that is to consider a Two-field model (quintessence and phantom). These models have settled out explicitly and named quintom models [51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69]. Quintom behavior (i.e., the  $w = -1$  crossing) has been investigated in the context of h-essence cosmologies [58, 59]; in the context of holographic dark energy [70, 71, 72, 73, 74]; inspired by string theory [61, 62, 63]; derived from spinor matter [64]; for arbitrary potentials [66, 67, 68, 69]; using isomorphic models consisting of three coupled oscillators, one of which carries negative kinetic energy (particularly for investigating the dynamical behavior of massless quintom)[338]. The crossing of the phantom divide is also possible in the context of scalar tensor theories [46, 47, 48, 49, 50] as well as in modified theories of gravity [75].

Alternative approaches to dark energy are also the so-called Extended Theory of Gravitation (ETG) and, in particular, higher-order theories of gravity (HOG) [76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100]. Such an approach can still be in the spirit of General Relativity Theory (GRT) since the only request the Hilbert-Einstein action should be generalized (by including non-linear terms in the Ricci curvature  $R$  and/or involving combinations of derivatives of  $R$  [101, 102, 103, 104]) asking for a gravitational interaction acting, in principle, in different ways in both cosmological [78, 79, 105, 106, 107, 108, 109] and astrophysical [105, 110] scales. In this case the field equations can be recast in a way that the higher order corrections are written as an energy-momentum tensor of geometrical origin describing an “effective” source term on the right hand side of the standard Einstein field equations [78, 79]. These models have been studied from the dynamical systems viewpoint in [100, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122].

Other alternatives are the models based on extra-dimensional theories, for example Randall-Sundrum braneworlds of type 2 (RS2). Randall-Sundrum braneworlds were first proposed in [123, 124]. In these references was proved that for non-factorizable geometries in five dimensions there exists a single massless bound state confined in a domain wall or three-brane. This bound state is the zero mode of the Kaluza-Klein dimensional

reduction and corresponds to the four-dimensional graviton. The RS2 model, was proposed as an alternative mechanism to the Kaluza-Klein compactifications [123], have been intensively studied in the last years, among other reasons, because its appreciable cosmological impact in the inflationary scenario [125, 126, 127]. The setup of the model start with the particles of the standard model confined in a four dimensional hypersurface with positive tension embedded in a 5-dimensional bulk with negative cosmological constant. It is well-known that the cosmological field equations on the brane are essentially different from the standard 4-dimensional cosmology. Friedmann-Robertson branes with an scalar field trapped on it have been investigated widely in the literature. In [128] was investigated the dynamics of a scalar field with constant and exponential potentials. These results were extended to a wider class of self-interaction potential in [129] using a method proposed in [130] supporting the idea that this scenario modifies gravity only at very high energy/short scales (UV modifications only) having an appreciable impact on primordial inflation but does not affecting the late-time dynamics of the Universe unless if the energy density of the matter trapped in the brane increase at late times [131]. Extra-dimensional alternatives are also the so-called Hořava-Lifshitz cosmology which is a power-counting renormalizable, ultra-violet (UV) complete theory of gravity [132, 133, 134, 135]. This theory as an infrared (IR) fixed point, namely General Relativity.

In [136] were investigated coupled dark energy models. There was proved that for coupling functions and potentials of desired differentiability class, the scalar field is unbounded to the past, but a set of zero-measure. There was devised a dynamical system appropriated to describe the past asymptotic dynamics, allowing to classify scaling solutions. There were presented asymptotic expansion rates for the cosmological solutions near the initial singularity, extending previous results. In [137] was investigated a cosmological model based in STT (and, therefore, related by conformal transformations with  $F(R)$ -theories) where it is considered a scalar field coupled to matter, and radiation is included. There was proved that the equilibrium points corresponding to the nonnegative local minimums of the potential (associated with cosmological de Sitter solutions) are asymptotically stable. In the same way as in [136], in [137] we prove that the scalar field is unbounded towards the past, but a set of zero-measure. A dynamical system was devised to investigate the dynamics near the initial singularity, obtaining in that regime: radiation-dominated scaling solutions; power-law inflationary scalar field dominated solutions; matter-kinetic-radiation scaling solutions; matter-potential-radiation scaling solutions. There were investigated the important examples of modified gravity  $F(R) = R + \alpha R^2$  (quadratic gravity) and  $F(R) = R^n$  gravity. In this book several of these results will be rigourously-established and also improved from the mathematical viewpoint.

Concerning the cosmological evolution of quintom models, this topic was investigated in [56] and [57, 60], from the dynamical systems viewpoint, by considering exponential

potentials. The difference between [56] and [57, 60] is that in the second case the potential function also accounts for the interaction between the conventional scalar field and the phantom field. In [57] it had been proven that in the absence of interactions, the phantom dominated solution should be the attractor of the system and that the interaction does not affect its attractor behavior. In [60] the case in which the interaction term dominates against the mixed terms of the potential, was studied. It was proven there, that the hypothesis in [57] is correct only in the cases in which the existence of the phantom phase excludes the existence of scaling attractors (in which the energy density of the quintom field and the energy density of DM are proportional). Some of this results were extended in [66], for arbitrary potentials. There it was settled down under what conditions on the potential it is possible to obtain scaling regimes. It was proved there, that for arbitrary potentials having asymptotic exponential behavior, scaling regimes are associated to the limit where the scalar fields diverge. Also it has been proven that the existence of phantom attractors in this framework is not generic and consequently the corresponding cosmological solutions lack the big rip singularity. In the reference [138] it is presented an exhaustive review of the quintom and quinstant paradigms.

In the reference [139] was investigated a scalar field with arbitrary potential trapped in a RS2 model. There were obtained sufficient conditions for the asymptotic stability of de Sitter solution and for the stability of scaling solutions as well as for the stability of the scalar-field dominated solution extending the results in [140] to the higher dimensional framework. In [139] was proved, also, the non-existence of late time attractors with 5D-modifications. A fact that correlates with a transient primordial inflation. The natural extension of the analysis in [139] is to consider a Bianchi I brane. Bianchi I models are the minimal extension of the FRW metric to the anisotropic framework. Homogenous but anisotropic geometries are well-known [141, 142]. Bianchi I, Bianchi III, and Kantowski-Sachs can be a very good representation for the homogeneous but anisotropic universe. They were investigated in the framework of  $f(R)$  cosmology from both numerical and analytical viewpoint also incorporating the matter content (see [100] and the references therein). The evolution of cosmological braneworld models were investigated, for instance, in [143, 144, 145, 146]. In [145] it is presented a systematic analysis of FRW, Bianchi I and Bianchi V metrics in these scenarios. There it is discussed the changes in the structure of the phase space with respect the general-relativistic case. In [147] it is studied the dynamics of a BI brane and it is showed that the high energy effects from extra dimensional gravity removes the anisotropic behavior near the initial singularity which is found in general relativity. For a Bianchi I-RS2 brane, it is possible to obtain isotropic late-time attractors compatibles with accelerated expansion for general classes of self-interacting potentials for a wide region in the parameter space (work in progress). In this way, the universe isotropizes towards the future, irrespectively of the initial anisotropy degree.

Following this line of reasoning, in [100], are investigated  $F(R) = R^n$  theories in anisotropic Kantowski-Sachs (KS) metrics. In this scenario the universe at late times can result to a state of accelerating expansion, and additionally, for a particular  $n$ -range ( $2 < n < 3$ ) it exhibits phantom behavior. Additionally, the universe has been isotropized, independently of the anisotropy degree of the initial conditions, and it asymptotically becomes flat. The fact that such features are in agreement with observations [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] is a significant advantage of the model. Moreover, in the case of radiation ( $n = 2$ ,  $w = 1/3$ ) the aforementioned stable solution corresponds to a de-Sitter expansion, and it can also describe the inflationary epoch of the universe. Note that at first sight the above behavior could be ascribed to the cosmic no-hair theorem [148], which states that a solution of the cosmological equations, with a positive cosmological constant and under the perfect-fluid assumption for matter, converges to the de Sitter solution at late times. However, we mention that such a theorem holds for matter-fluids less stiff than radiation, but more importantly it has been elaborated for General Relativity [149], without a robust extension to higher order gravitational theories [150]. In [100] were extracted the results without relying at all on the cosmic no-hair theorem, which is a significant advantage of the analysis. Apart from the above behavior, in this scenario the universe has a large probability to remain in a phase of (isotropic or anisotropic) decelerating expansion for a long time, before it will be attracted by the above global attractor at late times, and this acts as an additional advantage of the model, since it is in agreement with the observed cosmological behavior. The Kantowski-Sachs anisotropic  $R^n$ -gravity can also lead to contracting solutions, either accelerating or decelerating, which are not globally stable. Thus, the universe can remain near these states for a long time, before the dynamics remove it towards the above expanding, accelerating, late-time attractors. One of the most interesting behaviors is the possibility of the realization of the transition between expanding and contracting solutions during the evolution. That is, the scenario at hand can exhibit the cosmological bounce or turnaround. Additionally, there can also appear an eternal transition between expanding and contracting phases, that is we can obtain cyclic cosmology. These features can be of great significance for cosmology, since they are desirable in order for a model to be free of past or future singularities. In summary, anisotropic  $R^n$ -gravity has a very rich cosmological behavior, and a large variety of evolutions and late-time solutions, compatible with observations, that leads to radically different implications comparing to the simple isotropic scenarios.

In [151] was performed a detailed phase-space analysis of Hořava-Lifshitz cosmology, with and without the detailed-balance condition. Under detailed-balance the universe can reach a bouncing-oscillatory state at late times, in which dark-energy, behaving as a simple cosmological constant, is dominant. In this book, we use the Center Manifold Theory to investigate the stability of the de Sitter solution when detailed-balance condition is re-

laxed. In this case (where the detailed-balance condition is relaxed) the universe reaches an eternally expanding, dark-energy-dominated solution, with the oscillatory state preserving also a small probability. Since the phase space of Hořava-Lifshitz is, in general, non-compact, in this book we complement the results in [151] by performing a Poincaré compactification process in order to investigate the dynamics at infinity. This allows to construct a global phase space containing all the cosmological information.

Finally, in [152] was investigated several varying-mass dark-matter particle models in the framework of phantom cosmology. It is examined there whether there exist late-time cosmological solutions, corresponding to an accelerating universe and possessing dark energy and dark matter densities of the same order. Imposing exponential or power-law potentials and exponential or power-law mass dependence, the coincidence problem cannot be solved or even alleviated. Thus, if dark energy is attributed to the phantom paradigm, varying-mass dark matter models cannot fulfill the basic requirement that led to their construction. However, for completeness, in this book we use the Center Manifold Theory to analyze the stability of the non-hyperbolic fixed points in the phase space of dark-matter particle models in the framework of phantom cosmology. Basically, we use these cosmological models as examples of how to apply the Center Manifold Theory in cosmology. Also, in this book we perform a Poincaré compactification process allowing to construct a global phase space containing all the cosmological information in both finite and infinite regions.

Every one of these theoretical frameworks allows to explain several features of our dynamical universe. However, in general they are not valid to describe all the cosmic evolution. Following one or another direction, the systematic way to examine all possible cosmological behaviors of a particular model is to use dynamical systems techniques. Such an approach allows to by-pass the high nonlinearities and order of the cosmological equations (particularly, in the metric approach,  $F(R)$  models gives fourth order differential equations) which prevents a complete analytical treatment, obtaining a good global qualitative dynamics of the models under investigation.

Therefore, the proposition and validation of a cosmological model to explain the accelerated expansion of the universe is an open topic of discussion nowadays, which is verified in the variety of models proposed, without arriving to a definitive proposal yet. All these facts, explained before, justify the present investigation.

According to that, our investigation problem is related to has far the dynamical systems studies have contributed to the understanding of the evolution of the early- and late-time universe?; and more precisely, about if it is possible to apply the theory of dynamical systems to select among the cosmological paradigm, those models with proper past and future attractors.

This it is concreted in the following investigation question: How can we determine the asymptotic behavior of typical cosmological solutions with scalar fields, combining

topological, analytical and numerical methods of investigation?

Having examined the bibliography and having elaborated the theoretical framework, we formulate as investigation hypothesis that it is possible to obtain information about the local properties of the flow associated to an autonomous system of ordinary differential equations (ODEs), using qualitative techniques and considering a proper normalization and a proper parametrization only demanding good differentiability and integrability conditions for the input functions of the models.

Our general objective will be analyze, using a combination of topological, analytical and numerical techniques, the phase space of our investigation objects.

In order to fulfill our general objective, we have traced the specific objectives:

1. Obtaining all possible asymptotic behaviors for a quintessence field non-minimally coupled to matter including or not radiation, based on a STT.
2. Obtaining all possible asymptotic behaviors for mass-varying dark matter-particles in the framework of phantom cosmologies; and analyzing the viability of them in order to solve the coincidence problem (why the energy densities of dark matter and of dark energy are comparable in order the magnitude today?)
3. Obtaining all possible asymptotic behaviors for Hořava-Lifshitz cosmologies with and without detailed-balance.
4. Obtaining all possible asymptotic behaviors for the so-called Cardassian cosmologies.

The book is organized as follows. In chapter 1 we give a bird's eye view on cosmology and cosmological problems. Chapter 2 is devoted to a brief review on some results and useful tools from the qualitative theory of dynamical systems providing the theoretical basis for the qualitative study of concrete cosmological models. Chapters 3, 4, 5 and 6 are devoted to our main results. In these chapters are extended and settled in a substantially different, more strict mathematical language, several results obtained by one of us in [136, 137, 152, 151]. In chapter 6, we provide a different approach to the subject discussed in [452].



# Chapter 1

## A bird's eye view on cosmology and cosmological problems

### 1.1 Cosmological models

A cosmological model represents the Universe in a particular scale [153]. Within the most commonly accepted vision in Cosmology, it is assumed that the geometry of the universe at the large scale is described by General Relativity (see [141, 154, 155, 156]).

Each cosmological model is defined specifying ([157, 158]):

- (i) the *spacetime geometry*, represented in an averaged scale by the metric tensor  $g_{\alpha\beta}(x^\gamma)$ . Due the requirement of the compatibility with astrophysical observations, the metric of a cosmological model should have as a regular limit one of the Robertson- Walker ('RW') geometries; or at least, they should have properties compatibles with the observations inferred for the cosmological epoch that they are intended to describe;
- (ii) the *matter content*, represented in an averaged scale, and its *physical behavior*. The physical behavior of matter is described specifying the energy-momentum tensor of each matter component, their governing equations, the thermodynamical state equations and the interaction terms. Ideally, they should have a plausible physical interest (explaining the matter content of the universe from earlier epochs to the present, including the majority of the physical interactions described up to date); and
- (iii) the *interaction between matter and geometry* –how the matter determines geometry, and at the same time how geometry influences matter [159]–. In general it is assumed that this relation is given by the *Einstein's field equations of the gravitational*

field (EFEs) <sup>1</sup>

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta} - \Lambda g_{\alpha\beta} , \quad (1.1)$$

(here we have considered the possibility to introduce a non-null cosmological constant  $\Lambda$ ).  $R$  denotes the trace  $g^{\alpha\beta} R_{\alpha\beta}$ . Assuming that the *cosmological constant*  $\Lambda$  satisfies the relation  $\nabla_\alpha \Lambda = 0$ , i.e., if it is constant in time and in space, EFEs guaranteed the conservation of the total energy-momentum tensor through the *twice contracted Bianchi identities*,

$$\nabla_\beta G^{\alpha\beta} = 0 \quad \Rightarrow \quad \nabla_\beta T^{\alpha\beta} = 0 , \quad (1.2)$$

These three ingredients determine the combined evolution of geometry and the matter on it. The description should be sufficiently complete to determine:

- (iv) the *observational relations* predicted by the model for both discrete sources <sup>2</sup>, the cosmic microwave background radiation and the light-element abundances from nucleosynthesis in the early universe; implying a well-established theory for the *structure formation* in small and large physical scales.

Practically, as in any modelling of physical phenomena, we should compromise the model complexity in order to obtain the desired results. When referred to a cosmological model, this means assuming the existence of some symmetries. The usual hypotheses to describe the matter content are a combination of sources of any of the following types:

- (v) a fluid with a well-motivated physical state equation: for example, a perfect fluid with an specific equation of state;
- (vi) a mixture of fluids, usually with different 4-velocities;
- (vii) a set of particles represented by a kinetic theory description;
- (viii) a scalar field  $\phi$  (or several of them), with a given self-interacting potential  $V(\phi)$ .

In the phenomenological fluid description of a matter source, the standard decomposition of the energy-momentum tensor  $T_{\alpha\beta}$  with respect to a time-like vector field  $\vec{u}$  is given by:

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + 2q_{(\alpha} u_{\beta)} + p(g_{\alpha\beta} + u_\alpha u_\beta) + \pi_{\alpha\beta} , \quad (1.3)$$

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<sup>1</sup>Hereafter we employ units in which  $c = 1 = 8\pi G/c^2$ . Thus, all geometrical variables should have physical dimensions that are integer powers of [length].

<sup>2</sup>Observations of discrete sources (primarily galaxies, radio sources, infrared sources and quasars) provide information about the structure of the universe in the galactic epoch (say redshifts  $z \lesssim 5$ ). These observations can be grouped into three classes, *number count surveys*, that provide direct evidence concerning isotropy about our position, *redshift surveys*, that provide information about the inhomogeneities in the distribution of galaxies, and *peculiar velocity surveys*, that describe deviations from a uniform Hubble flow (see [1], pp 65-67).

where  $\rho$  denotes the relativistic energy density relative to  $u^\alpha$ ,  $p$  is the isotropic pressure,  $q^\alpha$  is the momentum density, and  $\pi_{\alpha\beta}$  is the trace-free anisotropic pressure. We have  $q_\alpha u^\alpha = 0$ ,  $\pi_{\alpha\beta} u^\beta = 0$ ,  $\pi_\alpha^\alpha = 0$ ,  $\pi_{\alpha\beta} = \pi_{\beta\alpha}$ . These quantities have to be related by appropriate thermodynamical equations of state in order to close the system of equations. These should provide a coherent representation of the underlying physics to the geometrical spacetime fluid scenario.

In this book we restrict ourselves to a non-tilted perfect fluid and to a scalar field. These matter sources admits, respectively, the following decompositions.

1. A *non-tilted perfect fluid* is described by its 4-velocity  $\vec{u}$ , its energy density  $\rho$  and pressure  $p$ , with barotropic equation of state  $p = p(\rho)$ . The energy-momentum tensor is

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + p(g_{\alpha\beta} + u_\alpha u_\beta), \quad u_\alpha u^\alpha = -1, \quad (1.4)$$

We will usually work with an equation of state  $p = (\gamma - 1)\rho$ , where  $\gamma$  is a constant. From a physical point of view, the most important values are  $\gamma = 1$  (dust) and  $\gamma = 4/3$  (radiation), specially for the early universe. The value  $\gamma = 0$  corresponds to a cosmological constant and the value  $\gamma = 2$  is occasionally considered corresponding to a “stiff” fluid. We assume that  $\gamma$  satisfies  $0 \leq \gamma \leq 2$ , in order to fulfill the energy requirements. The values  $\gamma > 2$  corresponds to fluids with supra-luminous propagation.

2. A canonical *scalar field* is described by the energy-momentum tensor

$$T_{\alpha\beta} = \nabla_\alpha \phi \nabla_\beta \phi - \left[ \frac{1}{2} \nabla_\gamma \phi \nabla^\gamma \phi + V(\phi) \right] g_{\alpha\beta}, \quad (1.5)$$

where the potential  $V(\phi)$  has to be specified. If  $\nabla_\alpha \phi$  is time-like, we can define a unit time-like vector field  $\vec{u}$  normal to the surfaces  $\phi = \text{const.}$  :

$$u^\alpha = \frac{\nabla^\alpha \phi}{(-\nabla_\gamma \phi \nabla^\gamma \phi)^{\frac{1}{2}}}.$$

Then,  $T_{\alpha\beta}$  has the algebraic form of a perfect fluid with

$$\rho = -\frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi + V(\phi), \quad p = -\frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi - V(\phi).$$

The equation of motion for the scalar field is the Klein-Gordon equation

$$\nabla^\alpha \nabla_\alpha \phi - V'(\phi) = 0,$$

which is a consequence of (1.5) and the conservation equation (1.2).

### 1.1.1 Homogeneity and isotropy: the Robertson-Walker metric

The current astrophysical observations points that our observable universe is homogeneous and isotropic at a great accuracy [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], leading the large majority of cosmological works to focus on homogeneous and isotropic geometries. Isotropy is the claim that the universe looks the same in all directions. Direct evidence for this comes from the smoothness of the Temperature of the CMB. Homogeneity is the claim that the universe looks the same at every point. We may therefore approximate the universe as a spatially homogeneous and isotropic three-dimensional space which may expand (or, in principle, contract) as a function of time. The metric on such a spacetime is necessarily the Robertson-Walker form.

Spatial isotropy implies spherical symmetry. Choosing a point as an origin, and using coordinates  $(x, \vartheta, \varphi)$  around this point, the spatial line element must take the form

$$d\sigma^2 = dr^2 + f^2(r) (d\theta^2 + \sin^2 \theta d\varphi^2) , \quad (1.6)$$

where  $f(r)$  is a real function, which, is the metric is to be nonsingular at the origin , obeys the law  $f(r) \sim r$  as  $r \rightarrow 0$ . It is proved that the more general metric of a spacetime consistent with homogeneity and isotropy is

$$ds^2 = -dt^2 + a^2(t) [d\rho^2 + f^2(\rho) (d\theta^2 + \sin^2 \theta d\phi^2)] , \quad (1.7)$$

where the three possibilities for  $f(\rho)$  are to be

$$f(\rho) = \{\sin(\rho), \rho, \sinh(\rho)\} . \quad (1.8)$$

This is a purely geometric fact, independent of the details of general relativity [160]. We have used spherical polar coordinates  $(\rho, \theta, \phi)$ , since spatial isotropy implies spherical symmetry about every point. The time coordinate  $t$ , which is the proper time as measured by a comoving observer (one at constant spatial coordinates), is referred to as cosmic time, and the function  $a(t)$  is called the scale factor.

There are two other useful forms for the RW metric. First, a simple change of variables in the radial coordinate yields

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dx^2}{1 - kx^2} + x^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right] , \quad (1.9)$$

where

$$k = \begin{cases} +1 & \text{si } f(\rho) = \sin(\rho) \\ 0 & \text{si } f(\rho) = \rho \\ -1 & \text{si } f(\rho) = \sinh(\rho) \end{cases} . \quad (1.10)$$

Geometrically,  $k$  describes the curvature of the spatial sections (slices at constant cosmic time).  $k = +1$  corresponds to positively curved spatial sections (locally isometric to 3-spheres);  $k = 0$  corresponds to local flatness, and  $k = -1$  corresponds to negatively curved (locally hyperbolic) spatial sections. These are all local statements, which should be expected from a local theory such as GR [160].

A second coordinate change, which may be applied to either (1.7) or (1.9), is to transform to a *conformal time*,  $\tau$ , via

$$\tau(t) \equiv \int^t \frac{dt'}{a(t')} . \quad (1.11)$$

Applying this to (1.9) yields

$$ds^2 = a^2(\tau) \left[ -d\tau^2 + \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] , \quad (1.12)$$

where we have written  $a(\tau) \equiv a[t(\tau)]$  as is conventional. The conformal time does not measure the proper time for any particular observer, but it does simplify some calculations.

A particular useful quantity to define from the scale factor is the *Hubble parameter* (sometimes called Hubble constant), given by

$$H \equiv \frac{\dot{a}}{a} . \quad (1.13)$$

The Hubble parameter relates how fast the most distant galaxies are receding from us to their distance from us via Hubble's law  $v \simeq Hd$ .

### 1.1.2 Dynamics: the Friedmann equations

As mentioned, the RW metric is a purely kinematic consequence of requiring homogeneity and isotropy of our spatial sections. We next turn to the dynamics, in form of differential equations governing the evolution of the scale factor  $a(t)$ . These will come from applying Einstein's equations ,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} \quad (1.14)$$

to the RW metric.

It is often to adopt the perfect fluid form for the energy-momentum tensor of cosmological matter. This form is

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} , \quad (1.15)$$

where  $U^\mu$  is the fluid 4-velocity,  $\rho$  is the energy density in the rest frame of the fluid and  $p$  is the pressure in that same frame. The pressure is necessarily isotropic for consistency with RW metric. Similarly, fluid elements will be comoving in the cosmological rest frame, so that the normalized 4-velocity in the coordinates of (1.9) will be

$$U^\mu = (1, 0, 0, 0) . \quad (1.16)$$

Therefore, the energy-momentum tensor takes the form

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & pg_{ij} \end{pmatrix} \quad (1.17)$$

where  $g_{ij}$  represents the spatial metric (including the factor of  $a^2$ ).

Armed with this simplified description for matter, we are now ready to apply Einstein's equation (1.14) to cosmology. Using (1.9) and (1.15), we obtain two equations. The first is known as the Friedmann equation,

$$H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \sum_i \rho_i - \frac{k}{a^2} , \quad (1.18)$$

where an overdot denotes a derivative with respect to cosmic time  $t$  and  $i$  indexes all different types of energy in the universe. This equation is a constraint equation, in the sense that we are not allowed to freely specify the time derivative  $\dot{a}$ ; it is determined in terms of the energy density and curvature. The second equation, which is an evolution equation, is

$$\frac{\ddot{a}}{a} + \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 = -4\pi G \sum_i p_i - \frac{k}{2a^2} . \quad (1.19)$$

It is often useful to combine (1.18) and (1.19) to obtain the *acceleration equation*

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_i (\rho_i + 3p_i) . \quad (1.20)$$

In fact, if we know the magnitudes and evolutions of the different energy density components  $\rho_i$ , the Friedmann equation (1.18) is sufficient to solve for the evolution uniquely. The acceleration equation is conceptually useful, but rarely invoked in calculations [160].

The Friedmann equation relates the rate of increase of the scale factor, as encoded by the Hubble parameter, to the total energy density of all matter in the universe. We may use the Friedmann equation to define, at any given time, a critical energy density,

$$\rho_c \equiv \frac{3H^2}{8\pi G} , \quad (1.21)$$

for which the spatial sections must be precisely flat ( $k = 0$ ). We then define the density parameter

$$\Omega_{\text{total}} \equiv \frac{\rho}{\rho_c}, \quad (1.22)$$

which allows us to relate the total energy density in the universe to its local geometry via

$$\begin{aligned} \Omega_{\text{total}} > 1 &\Leftrightarrow k = +1 \\ \Omega_{\text{total}} = 1 &\Leftrightarrow k = 0 \\ \Omega_{\text{total}} < 1 &\Leftrightarrow k = -1. \end{aligned} \quad (1.23)$$

It is often convenient to define the fractions of the critical energy density in each different component by

$$\Omega_i = \frac{\rho_i}{\rho_c}. \quad (1.24)$$

Energy conservation is expressed in GR by the vanishing of the covariant divergence of the energy-momentum tensor,

$$\nabla_\mu T^{\mu\nu} = 0. \quad (1.25)$$

Applying this to our assumptions –the RW metric (1.9) and a perfect energy-momentum tensor (1.15)– yields to a single energy-conservation equation,

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (1.26)$$

This equation is actually not independent of the Friedmann and acceleration equations, but it is required for consistency. It implies that the expansion of the universe (as specified by  $H$ ) can lead to local changes in the energy density. Note that there is no notion of conservation of “total energy”, as energy can be interchanged between matter and the spacetime geometry.

To close the equations we need to specify the equation of state. Within the fluid approximation we are using we may assume that the pressure is a single-valued function of the energy density  $p = p(\rho)$ . It is often convenient to define an equation of state (EoS) parameter,  $w$ , by

$$p = w\rho. \quad (1.27)$$

Several sources of cosmological matter satisfies this relation with  $w$  constant. For example,  $w = 0$  corresponds to pressureless matter, or dust – any collection of massive non-relativistic particles would qualify. Similarly,  $w = 1/3$  corresponds to a gas of radiation, whether it be actual photons or other highly relativistic species.

A constant value of  $w$  leads to a great simplification in solving our equations. In particular, using (1.26), we see that the energy density evolves with the scale factor according

to

$$\rho(a) \propto \frac{1}{a(t)^{3(1+w)}} . \quad (1.28)$$

We have not included a cosmological constant  $\Lambda$  in the gravitational equations. This is because it is equivalent to treat any cosmological constant as a component of the energy density in the universe. In fact, adding a cosmological constant  $\Lambda$  to Einstein's equation is equivalent to including an energy-momentum tensor of the form

$$T_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} . \quad (1.29)$$

This is simply a perfect fluid with energy momentum tensor (1.15) with

$$\begin{aligned} \rho_\Lambda &= \frac{\Lambda}{8\pi G} \\ p_\Lambda &= -\rho_\Lambda , \end{aligned} \quad (1.30)$$

so that the EoS parameter is

$$w_\Lambda = -1 . \quad (1.31)$$

This implies that the energy density is a constant,

$$\rho_\Lambda = \text{constant} . \quad (1.32)$$

Thus, this energy is constant throughout spacetime; we say that the cosmological constant is equivalent to *vacuum energy*.

Similarly, it is sometime useful to consider any non-null curvature as yet another component of the cosmological energy budget, obeying

$$\begin{aligned} \rho_k &= -\frac{3k}{8\pi G a^2} \\ p_k &= \frac{k}{8\pi G a^2} , \end{aligned} \quad (1.33)$$

so that

$$w_k = -1/3 . \quad (1.34)$$

It is not an energy density, of course;  $\rho_k$  is simply a convenient way to keep track of how much energy density is lacking, in comparison to a flat universe.

### 1.1.3 Flat universes

It is much easier to find exact solution to the cosmological equations for  $k = 0$ . Fortunately to us, nowadays we can appeal to more than mathematical simplicity to make this choice. The modern cosmological observations, particularly precision measurements of



Table 1.1: A summary of the behaviors of the most important sources of energy density in cosmology. The behavior of the scale factor applies to the case of a flat universe; the behavior of the energy densities is perfectly general.

Type of energy	$\rho(a)$	$a(t)$
Dust	$a^{-3}$	$t^{2/3}$
Radiation	$a^{-4}$	$t^{1/2}$
Cosmological constant	constant	$e^{Ht}$

the CMB, show the universe today to be extremely spatially flat.

In the case of flat spatial sections and for constant equation of state  $w$ , we can exactly solve the Friedmann equation (1.28) to obtain

$$a(t) = a_0 \left( \frac{t}{t_0} \right)^{2/3(1+w)}, \quad (1.35)$$

where  $a_0$  is the scale factor today, unless  $w = -1$ , in which case we obtain  $a(t) \propto e^{Ht}$ . Applying this result to the more commonly used matter sources we obtain the results showed in table 1.1.

### 1.1.4 Including curvature

It is true that we know observationally that the universe today is flat to a high degree of accuracy. However, it is instructive, and useful when considering early cosmology, to consider how the solutions we have already identified change when curvature is included. We discuss some examples from [160] by working in terms on the conformal time  $\tau$ .

Let us first consider models in which the energy density is dominated by matter ( $w = 0$ ). In terms of conformal time, Einstein equations become

$$\begin{aligned} 3(k + h^2) &= 8\pi G\rho a^2 \\ k + h^2 + 2h' &= 0, \end{aligned} \quad (1.36)$$

where a prime denotes derivative with respect to the conformal time and  $h(\tau) \equiv a'/a$ . These equations are then easily solved for  $h(\tau)$  giving

$$h(\tau) = \begin{cases} \cot(\tau/2) & k = 1 \\ 2/\tau & k = 0 \\ \coth(\tau/2) & k = -1 \end{cases}. \quad (1.37)$$

This then yields

$$a(\tau) \propto \begin{cases} 1 - \cos(\tau) & k = 1 \\ \tau^2/2 & k = 0 \\ \cosh(\tau) - 1 & k = -1 \end{cases} . \quad (1.38)$$

One may use this to derive the connection between cosmic time and conformal time, which here is

$$t(\tau) \propto \begin{cases} \tau - \sin(\tau) & k = 1 \\ \tau^3/6 & k = 0 \\ \sinh(\tau) - \tau & k = -1 \end{cases} . \quad (1.39)$$

Next we consider models dominated by radiation ( $w = 1/3$ ). In terms of conformal time, the Einstein equations become

$$\begin{aligned} 3(k + h^2) &= 8\pi G\rho a^2 \\ k + h^2 + 2h' &= -\frac{8\pi G\rho}{3}a^2 . \end{aligned} \quad (1.40)$$

Solving as we did above yields

$$h(\tau) = \begin{cases} \cot(\tau) & k = 1 \\ 1/\tau & k = 0 \\ \coth(\tau) & k = -1 \end{cases} , \quad (1.41)$$

$$a(\tau) \propto \begin{cases} \sin(\tau) & k = 1 \\ \tau & k = 0 \\ \sinh(\tau) & k = -1 \end{cases} , \quad (1.42)$$

and

$$t(\tau) \propto \begin{cases} 1 - \cos(\tau) & k = 1 \\ \tau^2/2 & k = 0 \\ \cosh(\tau) - 1 & k = -1 \end{cases} . \quad (1.43)$$

Is is straightforward to interpret these solution by examining the behavior of the scale factor  $a(\tau)$ ; the qualitative features are the same for matter- or radiation-domination. In both cases, the universes with positive curvature ( $k = +1$ ) expand from an initial singularity with  $a = 0$ , and latter recollapsing again. The initial singularity is the Big Bang, and the final singularity is sometimes called the Big-Crunch. The universes with zero or negative curvature begin at the Big-Bang and expand forever.

### 1.1.5 Geometry, destiny and dark energy

We know that in matter- or radiation-dominated universes with energy density greater than the critical would ultimately collapse, while those with less than the critical would

expand forever, with flat universes lying in the border between the two. For the case of purely dust-filled universes this is easily seen from (1.38) and (1.42).

The connection between geometry and destiny was quite reasonably as long as dust and radiation were the only types of energy relevant in the late universe.

In recent years it has become clear that the dominant component of energy density in the present universe is neither dust nor radiation, but rather is dark energy. This component is characterized by an equation of state parameter  $w < -1/3$ .

For simplicity let us focus on what happens if the only energy density in the universe is a cosmological constant, with  $w = -1$ . In this case the Friedmann equation may be solve for any value of the spatial curvature parameter. If  $\Lambda > 0$  then the solutions are

$$\frac{a(t)}{a_0} = \begin{cases} \cosh\left(\sqrt{\frac{\Lambda}{3}}t\right) & k = +1 \\ \exp\left(\sqrt{\frac{\Lambda}{3}}t\right) & k = 0 \\ \sinh\left(\sqrt{\frac{\Lambda}{3}}t\right) & k = -1 \end{cases}, \quad (1.44)$$

where we have encountered the case  $k = 0$  earlier. It is immediately clear that in the limit  $t \rightarrow \infty$ , all the solutions expand exponentially, independently of the spatial curvature. In fact, these solutions are exactly the same spacetime *de Sitter space*, just in different coordinate systems. The crucial point is that the universe clearly expands forever in these spacetimes, irrespective of the value of the spatial curvature. Note, however, that not all of the solutions in (1.44) actually cover all the de Sitter spacetime; the  $k = 0$  and  $k = -1$  solutions represent coordinate patches which only cover part of the manifold. For completeness, let us complete the description of spacetimes with a cosmological constant by considering the case  $\Lambda < 0$ . This spacetime is called *Anti-de Sitter space* (AdS) and it should be clear from the Friedmann equation that such a spacetime can only exist in a space with spatial curvature  $k = -1$ . The corresponding solution for the scale factor is

$$a(t) = a_0 \sin\left(\sqrt{-\frac{\Lambda}{3}}t\right). \quad (1.45)$$

Once again, this solution does not cover all of AdS.

### 1.1.6 Scalar fields and Dark Energy

Dark energy is one of the hottest topics in precision cosmology (see the recent reviews [23, 24, 25, 26, 138, 161]), with the cosmological constant being the more arguably and economical candidate. This proposal suffers, however, from the well-known problem of fine tuning. That is, there exists a (yet) unexplained discrepancy (close to 120 orders of magnitude) between the cosmological observed value and the value predicted by Quantum Field Theory. This problem has been baptized as the Cosmological Constant Problem.

Rather than dealing directly with the cosmological constant a number of alternatives routes have been proposed which skirt around this thorny issue. An incomplete list includes

- i) Quiescence with  $w \equiv p_{DE}/\rho_{DE} = \text{const.}$ , the cosmological constant  $\Lambda$  ( $w = -1$ ) is a special member of this class.
- ii) Quintessence models which are inspired by the simplest class of inflationary models of the early universe and employ a scalar field rolling down a potential  $V(\phi)$  to achieve acceleration. Quintessence potentials with  $V''V/(V')^2 \geq 1$  have the attractive property that the dark energy approaches a common evolutionary “tracker path” from a wide range of initial conditions.
- iii) K-essence which is characterized by a scalar field with a non-canonical kinetic energy. The most general scalar field action is a function of  $\phi$  and  $X \equiv -\frac{1}{2}(\nabla\phi)^2$ .
- iv) Tachyon fields with the more general Lagrangian  $\mathcal{L}_\phi = V(\phi)\sqrt{-\det(g_{\alpha\beta} + \partial_\alpha\phi\partial_\beta\phi)}$ .
- v) The Chaplygin gas model (CG) has the equation of state  $p \propto -1/\rho$  and evolves as  $\rho = \sqrt{A + B(1+z)^6}$  where  $z$  is the redshift,  $z = a(t_0)/a(t) - 1$ . It therefore behaves like dark matter at early times ( $z \gg 1$ ) and like the cosmological constant at late times. CG appears to be the simplest model attempting to unify DE and non-baryonic cold dark matter.
- vi) “Phantom” DE ( $w < -1$ ).
- vii) Oscillating DE.
- viii) Models with interactions between DE and DM.
- ix) Scalar-tensor DE models.
- x) Modified Gravity DE models in which the gravitational Lagrangian is changed from  $R$  to  $F(R)$  where  $R$  is the curvature scalar and  $F$  is an arbitrary function.
- xi) Dark energy driven by quantum effects.
- xii) Higher dimensional “braneworlds” models in which acceleration is caused by the leakage of gravity into extra dimensions.
- xiii) Holographic dark energy, etc.

Table 1.2: DE models based on scalar fields ( $X \equiv -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$ ).

Model	Lagrangian Density	Motion Equations.
Quintessence	$\mathcal{L}_\phi = -V(\phi) + X$	$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$
Tachyon	$\mathcal{L}_\phi = -V(\phi)\sqrt{1-2X}$	$\frac{\ddot{\phi}}{1-\phi^2} + 3H\dot{\phi} + \frac{1}{V}\frac{dV}{d\phi} = 0$
Phantom	$\mathcal{L}_\phi = -V(\phi) - X$	$\ddot{\phi} + 3H\dot{\phi} - \frac{dV}{d\phi} = 0$
K-essence	$\mathcal{L}_\phi = L(\phi, X)$ $L$ no lineal en $X$	$\left(\frac{\partial L}{\partial X} + 2X\frac{\partial^2 L}{\partial X^2}\right)\ddot{\phi} + \frac{\partial L}{\partial X}(3H\dot{\phi}) + \frac{\partial^2 L}{\partial\phi\partial X}\dot{\phi}^2 - \frac{\partial L}{\partial\phi} = 0$

In table (1.2) we enumerate some of the Lagrangian densities of the most common models for DE. Additionally we present the corresponding equations of motion for the scalar field in the case of homogeneous cosmologies.

### Quintessence

The most popular scalar field DE models are the quintessence field [27, 162, 163]. These models are described by a conventional scalar field  $\phi$  minimally coupled to gravity. An adequate choice of the interaction potential gives the late time acceleration of the universe. The action for quintessence is given by:

$$S = \int d^4x \sqrt{|g|} \mathcal{L}_\phi \quad (1.46)$$

(see definition of  $\mathcal{L}_\phi$  in table (1.2)).

The energy-momentum tensor of quintessence field (obtained by varying the action (1.46) with respect to  $g^{\alpha\beta}$  and having into account the identity  $\delta\sqrt{|g|} = -(1/2)\sqrt{|g|}g_{\alpha\beta}\delta g^{\alpha\beta}$ ) is given by:

$$T_{\alpha\beta} \equiv -\frac{2}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\alpha\beta}} = \partial_\alpha\phi\partial_\beta\phi - g_{\alpha\beta}\left[\frac{1}{2}g^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi + V(\phi)\right]. \quad (1.47)$$

In the spatially homogeneous case, the energy density and isotropic pressure are given respectively by  $\rho = -T_0^0 = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ ,  $p = T_i^i = \frac{1}{2}\dot{\phi}^2 - V(\phi)$ . We assume that  $w \equiv \frac{p}{\rho} \in [-1, -\frac{1}{3}]$ .

### Phantom

Recently it has been argued that (combined) astrophysical observations (from SNIa and CMB) might favour a DE component with EoS parameter  $\omega = p/\rho < -1$  [18, 19, 22, 26, 164, 165], where  $p$  is the pressure and  $\rho$  is the energy density of the fluid. Sources sharing this property violate the null dominant energy condition (NDEC)<sup>3</sup>.

<sup>3</sup>That NDEC-violating sources can occur has been argued decades ago, e.g., see references [169, 170, 171].

NDEC-violating sources have been investigated as possible DE candidates and have been called phantom components [166, 167].

Since NDEC prevents instability of the vacuum or propagation of energy outside the light cone, phantom models are intrinsically unstable <sup>4</sup>. Nevertheless, if thought of as effective field theories (valid up to a given momentum cutoff) these models could be phenomenologically viable [168].

Phantom DE can be described by an unconventional scalar field  $\phi$  minimally coupled to gravity. Its action is given by:

$$S = \int d^4x \sqrt{|g|} \mathcal{L}_\phi \quad (1.48)$$

(see definition of  $\mathcal{L}_\phi$  in table 1.2). The energy-momentum tensor is very similar to (1.47) with the identification

$$\partial_\alpha \phi \partial_\beta \phi \rightarrow -\partial_\alpha \phi \partial_\beta \phi. \quad (1.49)$$

Due to the unorthodox character of considering the wrong sign of the kinetic term we want to comment further on this particular. As already noted, the argument most used for considering phantom matter at classical level is linked with the idea that, at long distances, the field theory of the phantom particle is an effective theory whose ultraviolet completion is well defined and respects unitarity [173]. In effect, the divergent nature of the instability caused by a massive ghost excitations can be avoided only by imposing a Lorentz non-invariant momentum space cutoff such that the vacuum decay rate would be slow enough on cosmological time scales. In [173], the magnitude of the cutoff is estimated by requiring consistency with observational constraints. The authors found that the low-effective ghosts must originate from new physics far below the TeV scale. String theory, in particular is ruled out as possible source for effective ghosts [173]. A possibility, also anticipated in [173], is that the phantom field massive ghost excitations might come from a low-energy sector that is completely hidden from the standard model particles, except for gravitational couplings. Braneworlds, in particular Randall-Sundrum type 2 (RS2) proposed in the reference [174] could be such a source for effective ghosts required to hold the phantom dark energy hypothesis [45]. Besides, the effective phantom nature of the DE can be reinterpreted as arising from dynamical-screening of the brane cosmological constant in Dvali-Gabadadze-Porrati (DGP) braneworld model (proposed in [175]) with a standard brane cosmological constant [176].

#### The Big-Rip singularity

To the number of unwanted properties of a phantom component with “supernegative” EOS parameter  $\omega_{ph} = p_{ph}/\rho_{ph} < -1$ , we add the fact that its energy density  $\rho_{ph}$  increases in an expanding universe. <sup>5</sup> This property ultimately leads to a catastrophic (future)

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<sup>4</sup>Another very strange property of phantom universes is that their entropy is negative [172].

<sup>5</sup>Alternatives to phantom models to account for supernegative EOS parameter have been considered

big rip singularity that is characterized by divergences in the scale factor  $a$ , the Hubble parameter  $H$  and its time-derivative  $\dot{H}$  [178]. In fact, the Einstein's field equations for a flat FRW universe with line element  $ds^2 = -dt^2 + a^2(t) (dx^2 + d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$ , filled with a barotropic perfect fluid with EoS  $w = p/\rho$  ( $w$  const.) admits the solution  $H = \frac{2}{3(1+w)(t-t_0)}$ ,  $a(t) \propto (t-t_0)^{\frac{2}{3(1+w)}}$ ,  $\rho \propto a^{-3(1+w)}$ , where  $t_0$  is a constant and  $w \neq -1$ . Observe that for  $w < -1$  describe a contracting universe. However, these equations admits an expanding solution given by  $a(t) = (t_s - t)^{\frac{2}{3(1+w)}}$ , where  $t_s$  is a constant. This corresponds to a super-inflationary solution where the Hubble scalar and curvature increases with time respectively as  $H = \frac{n}{t_s - t}$ ,  $n = -\frac{2}{3(1+w)} > 0$ ,  $R = 6 \left( 2H^2 + \dot{H} \right) = \frac{6n(2n+1)}{(t_s - t)^2}$ . The scale factor diverge (and consequently, matter density) as  $t \rightarrow t_s$  (which corresponds to an infinitely large quantity in a finite proper time towards the future) provided  $w < -1$ . In that limit, the Hubble scalar and curvature become infinity. This situation has been referred as Big-Rip singularity. This is an inherent property of all generic phantom DE model [179]. A detailed study of the kinds of singularity might occur in phantom scenarios (including the big rip) has been the target of references [180, 181, 182].

This singularity is at a finite amount of proper time into the future but, before it is reached, the phantom energy will rip apart all bound structures, including molecules, atoms and nuclei. To avoid this catastrophic event (also called "cosmic doomsday"), some models and/or mechanisms have been invoked. In the references [183, 184, 185], for instance, it has been shown that this singularity in the future of the cosmic evolution might be avoided or, at least, made milder if quantum effects are taken into consideration. Instead, a suitable perturbation of de Sitter equation of state can also lead to classical evolution free of the big rip [186]. Gravitational back reaction effects [187] and scalar fields with negative kinetic energy term with self interaction potentials with maxima [168, 188] have also been considered. Another way to avoid the unwanted big rip singularity is to allow for a suitable interaction between the phantom energy and the background (DM) fluid [45, 189, 190].

#### Interacting Dark energy

Although experimental tests in the solar system impose severe restrictions on the possibility of non-minimal coupling between the DE and ordinary matter fluids [191], due to the unknown nature of the dark matter (DM) as part of the background, it is possible to have additional (non gravitational) interactions between the DE and the DM components, without conflict with the experimental data. Thus, if there is transfer of energy from the phantom component to the background fluid, it is possible to arrange the free parameters of the model, in such a way that the energy densities of both components decrease with time, thus avoiding the big rip [45, 189].

Since there is exchange of energy between the phantom and the background fluids, the energy is not conserved separately for each component. Instead, the continuity equation

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also. See, for instance, references [177].

for each fluid shows a source (interaction) term [45]:

$$\dot{\rho}_m + 3H(\rho_m + p_m) = Q, \quad (1.50)$$

$$\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = -Q, \quad (1.51)$$

where the dot accounts for derivative with respect to the cosmic time and  $Q$  is the interaction term. Note that the total energy density  $\rho_T = \rho_M + \rho_{DE}$  ( $p_T = p_M + p_{DE}$ ) is indeed conserved:  $\dot{\rho}_T + 3H(\rho_T + p_T) = 0$ . To specify the general form of the interaction term, the authors look at a scalar-tensor theory of gravity where the matter degrees of freedom and the scalar field are coupled in the action through the scalar-tensor metric  $\chi(\phi)^{-1}g_{ab}$  [45, 192]:

$$S_{ST} = \int d^4x \sqrt{|g|} \left\{ \frac{R}{2} - \frac{\epsilon}{2}(\nabla\phi)^2 - V(\phi) + \chi(\phi)^{-2} \mathcal{L}_m(\mu, \nabla\mu, \chi^{-1}g_{ab}) \right\}, \quad (1.52)$$

where  $\epsilon = \pm 1$  ( $\epsilon = -1$  for phantom models and  $\epsilon = 1$  for quintessence models),  $V(\phi)$  is the scalar field self-interaction potential,  $\chi(\phi)^{-2}$  is the coupling function,  $\mathcal{L}_m$  is the matter Lagrangian and  $\mu$  is the collective name for the matter degrees of freedom. It can be shown that, in terms of the coupling function  $\chi(\phi)$ , the interaction term  $Q$  in equations (1.50) and (1.51), can be written in the following form:

$$Q = \rho_M H \left[ a \frac{d(\ln \bar{\chi})}{da} \right], \quad (1.53)$$

where it is used the “reduced” notation  $\bar{\chi}(a) \equiv \chi(a)^{(3\omega_M-1)/2}$  and it has been assumed that the coupling can be written as a function of the scale factor  $\bar{\chi} = \bar{\chi}(a)$ . Comparing this with other interaction terms in the bibliography, one can obtain the functional form of the coupling function  $\bar{\chi}$  in each case. In the reference [189], for instance,  $Q = 3Hc^2(\rho_{ph} + \rho_M) = 3c^2 H \rho_M (r+1)/r$ , where  $c^2$  denotes the transfer strength and  $r \equiv \Omega_m/\Omega_{ph}$ . If one compares this expression with (1.53) one obtains the following coupling function:

$$\bar{\chi}(a) = \bar{\chi}_0 e^{3 \int \frac{da}{a} \left( \frac{r+1}{r} \right) c^2}, \quad (1.54)$$

where  $\bar{\chi}_0$  is an arbitrary integration constant. If  $c^2 = c_0^2 = \text{const.}$  and  $r = r_0 = \text{const.}$ , then  $\bar{\chi} = \bar{\chi}_0 a^{3c_0^2(r_0+1)/r_0}$ .

### The coincidence problem

Models with interaction between the phantom and the DM components are also appealing since the coincidence problem (why the energy densities of dark matter and dark energy are of the same order precisely at present?) can be solved or, at least, smoothed out by



considering a non-gravitational coupling between both dark sources [45, 193, 194, 195, 196, 197, 198, 199, 200].

It has been shown, in particular, that a suitable coupling, can produce scaling solutions. The way in which the coupling is approached is not unique. In reference [193, 194], for instance, the coupling is introduced by hand. In [196, 197, 198] the type of coupling is not specified from the beginning. Instead, the form of the interaction term is fixed by the requirement that the ratio of the energy densities of DM and quintessence has an stable fixed point during the evolution that solves the coincidence; in [196] a suitable interaction between the quintessence field and DM leads to a transition from the domination matter era to an accelerated expansion epoch in the model proposed in [196, 197, 198]. A model derived from the Dilaton is studied in [195]. In this model the coupling function is chosen as a Fourier expansion around some minimum of the scalar field.

It is interesting to discuss, in the general case, under which conditions the coincidence problem might be avoided in models with interaction among the components in the mixture. In this sense one expects that a regime with simultaneous non zero values of the density parameters of the interacting components is a singular point of the corresponding dynamical system, so the system lives in this state for a sufficiently long period of time and, hence, the coincidence does not arises.

For this purpose it is recommended to study the dynamics of the ratio function  $r$ :

$$r = \frac{\rho_M}{\rho_\phi} = \frac{\Omega_m}{\Omega_\phi}, \quad (1.55)$$

in respect to the time variable  $\tau \equiv \ln a$  (it is related to the cosmic time through  $d\tau = H dt$ ). The following generic evolution equation holds for  $r$ :

$$r' = \frac{\rho_M}{\rho_\phi} \left( \frac{\rho'_M}{\rho_M} - \frac{\rho'_\phi}{\rho_\phi} \right) = f(r), \quad (1.56)$$

where the prime denotes derivative in respect to  $\tau$ , and  $f$  is an arbitrary function (at least of class  $\mathcal{C}^1$ ) of  $r$ . One is then primarily interested in the equilibrium points of equation (1.56), i.e., those points  $r_{ei}$  at which  $f(r_{ei}) = 0$ . After that one expands  $f$  in the neighborhood of each equilibrium point;  $r = r_{ei} + \epsilon_i$ , so that, up to terms linear in the perturbations  $\epsilon_i$ :  $f(r) = (df/dr)_{r_{ei}}\epsilon_i + \mathcal{O}(\epsilon_i) \Rightarrow \epsilon'_i = (df/dr)_{r_{ei}}\epsilon_i$ . This last equation can be integrated to yield the evolution of the perturbations in time  $\tau$ :

$$\epsilon_i = \epsilon_{0i} \exp [(df/dr)_{r_{ei}}\tau], \quad (1.57)$$

where  $\epsilon_{0i}$  are arbitrary integration constants. It is seen from (1.57) that, only those perturbations for which:

$$(df/dr)_{r_{ei}} < 0, \quad (1.58)$$

decay with time  $\tau$ , and the corresponding equilibrium point is stable. The necessary condition to evade the coincidence problem is then given by the requirement that the point  $\rho_M/\rho_\phi = r_{ei} \lesssim 1$  be stable against small linear perturbations of the kind explained above.

If we take into account the conservation equations (1.50) and (1.51), and the definition of the interaction term  $Q$  given in equation (1.53), then, the function  $f$  can be given by the following expression:

$$f(r) = r [(\ln \bar{\chi})'(r+1) + 3(\gamma_\phi - \gamma_m)]. \quad (1.59)$$

Note that, for a model without interaction ( $(\ln \bar{\chi})' = 0$ ) and with a constant DE barotropic parameter  $\gamma_\phi = \gamma_{\phi,0}$  (consider, for simplicity, dust-like background fluid so that  $\gamma_m = 1$ );  $f(r) = 3(\gamma_{\phi,0} - 1)r$  and the only (stable) equilibrium point is the dark energy dominated solution  $r = 0 \Rightarrow \Omega_\phi = 1$ . In consequence the coincidence does arise in this case.

#### Varying-mass dark matter particles in the framework of phantom cosmologies

An equivalent approach to the interacting dark energy (for instance, inspired in the action (1.52)) is to assume that dark energy and dark matter sectors interact in such a way that the dark matter particles acquire a varying mass, dependent on the scalar field which reproduces dark energy [201]. This consideration allows for a better theoretical justification, since a scalar-field-dependent varying-mass can arise from string or scalar-tensor theories [202]. Indeed, in such higher dimensional frameworks one can formulate both the appearance of the scalar field (which is related to the dilaton and moduli fields) and its effect on matter particle masses (determined by string dynamics, supersymmetry breaking, and the compactification mechanism) [203]. In quintessence scenario, such varying-mass dark matter models have been explored in cases of linear [201, 203, 204, 205], power-law [206] or exponential [193] scalar-field dependence. The exponential case is the most interesting since, apart from solving the coincidence problem, it allows for stable scaling behavior, that is for a large class of initial conditions the cosmological evolution converges to a common solution at late times [193].

Let us construct a cosmological model where dark energy is attributed to a phantom field, in which the dark matter particles have a varying mass depending on this field. Throughout the work we consider a flat Robertson-Walker metric:

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2, \quad (1.60)$$

with  $a$  the scale factor and  $t$  the comoving time.

In the phantom cosmological paradigm the energy density and pressure of the phantom

scalar field  $\phi$  are:

$$\rho_\phi = -\frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (1.61)$$

$$p_\phi = -\frac{1}{2}\dot{\phi}^2 - V(\phi), \quad (1.62)$$

where  $V(\phi)$  is the phantom potential and the dot denotes differentiation with respect to comoving time. In such a scenario, the dark energy is attributed to the phantom field, and its equation of state is given by

$$w_{DE} \equiv w_\phi = \frac{p_\phi}{\rho_\phi}. \quad (1.63)$$

In varying-mass dark matter models the central assumption is that the dark-matter particles have a  $\phi$ -dependent mass  $M_{DM}(\phi)$ , while dark matter is considered as dust. Thus, for the dark matter energy density we have the standard definition

$$\rho_{DM} = M_{DM}(\phi) n_{DM}, \quad (1.64)$$

where  $n_{DM}$  is the number density of the dark-matter particles. As usual, in the case of FRW geometry, it is determined by the equation

$$\dot{n}_{DM} + 3Hn_{DM} = 0, \quad (1.65)$$

with  $H$  the Hubble parameter. Therefore, differentiating (1.64) and using (1.65) we obtain the evolution equation for  $\rho_{DM}$ , namely:

$$\dot{\rho}_{DM} + 3H\rho_{DM} = \frac{1}{M_{DM}(\phi)} \frac{dM_{DM}(\phi)}{d\phi} \dot{\phi} \rho_{DM}. \quad (1.66)$$

Obviously, in a case of  $\phi$ -independent dark-matter particle mass, we re-obtain the usual evolution equation  $\dot{\rho}_{DM} + 3H\rho_{DM} = 0$ . Therefore, we observe that the  $\phi$ -dependent mass reveals the interaction between dark matter and dark energy (that is the phantom field) sectors that lies behind it.

Since general covariance leads to total energy conservation, we deduce that the evolution equation for the phantom energy density will be:

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\frac{1}{M_{DM}(\phi)} \frac{dM_{DM}(\phi)}{d\phi} \dot{\phi} \rho_{DM}. \quad (1.67)$$

Thus,  $\frac{dM_{DM}(\phi)}{d\phi} \dot{\phi} < 0$  corresponds to energy transfer from dark matter to dark energy, while  $\frac{dM_{DM}(\phi)}{d\phi} \dot{\phi} > 0$  corresponds to dark energy transformation into dark matter.

Equivalently, using the definitions (1.61) and (1.62), the phantom evolution equation

can be written in field terms as:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\partial V(\phi)}{\partial \phi} = \frac{1}{M_{DM}(\phi)} \frac{dM_{DM}(\phi)}{d\phi} \rho_{DM}. \quad (1.68)$$

Finally, the system of equations closes by considering the Friedmann equations:

$$H^2 = \frac{\kappa^2}{3}(\rho_\phi + \rho_{DM}), \quad (1.69)$$

$$\dot{H} = -\frac{\kappa^2}{2}(\rho_\phi + p_\phi + \rho_{DM}), \quad (1.70)$$

where we have set  $\kappa^2 \equiv 8\pi G$ . Although we could straightforwardly include baryonic matter and radiation in the model, for simplicity reasons we neglect them.

Alternatively, one could construct the equivalent uncoupled model described by:

$$\dot{\rho}_{DM} + 3H(1 + w_{DM,eff})\rho_{DM} = 0 \quad (1.71)$$

$$\dot{\rho}_\phi + 3H(1 + w_{\phi,eff})\rho_\phi = 0, \quad (1.72)$$

where

$$w_{DM,eff} = -\frac{1}{M_{DM}(\phi)} \frac{dM_{DM}(\phi)}{d\phi} \frac{\dot{\phi}}{3H} \quad (1.73)$$

$$w_{\phi,eff} = w_\phi + \frac{1}{M_{DM}(\phi)} \frac{dM_{DM}(\phi)}{d\phi} \frac{\dot{\phi}}{3H} \frac{\rho_{DM}}{\rho_\phi}. \quad (1.74)$$

However, it is more convenient to introduce the “total” energy density  $\rho_{tot} \equiv \rho_{DM} + \rho_\phi$ , obtaining:

$$\dot{\rho}_{tot} + 3H(1 + w_{tot})\rho_{tot} = 0, \quad (1.75)$$

with

$$w_{tot} = \frac{p_\phi}{\rho_\phi + \rho_{DM}} = w_\phi \Omega_\phi, \quad (1.76)$$

where  $\Omega_\phi \equiv \frac{\rho_\phi}{\rho_{tot}} \equiv \Omega_{DE}$ . Obviously, since  $\rho_{tot} = 3H^2/\kappa^2$ , (1.75) leads to a scale factor evolution of the form  $a(t) \propto t^{2/(3(1+w_{tot}))}$ , in the constant  $w_{tot}$  case. However, at the late-time stationary solutions that we are studying in the present work,  $w_{tot}$  has reached to a constant value and thus the above behavior is valid. Therefore, we conclude that in such stationary solutions the condition for acceleration is just  $w_{tot} < -1/3$ .

In the reference [207] were investigated varying-mass dark matter models in scenarios where dark energy is attributed to a phantom field. By imposing exponential or power-law potentials and exponential or power-law mass dependence, was proved there that the coincidence problem cannot be solved or even alleviated if dark energy is attributed to the phantom paradigm. Apart from the case of an exponential potential with an exponentially-

dependent dark-matter particle mass, which possesses a relevant late-time (phantom) attractor, in all the other models we found that physical, well-motivated solutions have a very small chance to attract the universe at late times. In addition, in all the examined cases, solutions having  $\Omega_{\text{DE}}/\Omega_{\text{DM}} \approx \mathcal{O}(1)$  were not relevant attractors at late times. Therefore, the coincidence problem cannot be solved or even alleviated in varying-mass dark matter particles models in the framework of phantom cosmology, in a radical contrast with the corresponding quintessence case. However, there exists few interacting phantom models that solves the coincidence <sup>6</sup> but paying the price of introducing new problems such is the justification of a non-trivial, almost tuned, sequence of cosmological epochs. This is the case of [45], where it is described a methodology to generate solutions free of the Big-Rip singularity that solves the coincidence.

### Scalar-tensor theories

Scalar-tensor theories (STT) of gravity [208, 209, 210, 211, 212, 213, 214] can be supported by fundamental physical theories like superstring theory [215].

BDT is the first prototype of STT. In BDT, a scalar field,  $\chi$ , acts as the source for the gravitational coupling with a varying Newtonian 'constant'  $G \sim \chi^{-1}$ . It is worthy to mention that BDT survive several observational tests including Solar System tests [216] and Big-Bang nucleosynthesis constraints [217, 218]. More general STT with a non-constant BD parameter  $\omega(\chi)$ , and non-zero self-interaction potential  $V(\chi)$ , have been formulated, and also survive astrophysical tests [191, 219, 220].

“Extended” inflation models [221, 222, 223, 224] use the Brans-Dicke theory (BDT) [208] as the correct theory of gravity, and in this case the vacuum energy leads directly to a powerlaw solution [225] while the exponential expansion can be obtained if a cosmological constant is explicitly inserted into the field equations [221, 226, 227].

The action for a general class of STT, written in the so-called Einstein frame (EF), is given by [192]:

$$S_{EF} = \int_{M_4} d^4x \sqrt{|g|} \left\{ \frac{1}{2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \chi(\phi)^{-2} \mathcal{L}_m(\mu, \nabla\mu, \chi(\phi)^{-1}g_{\alpha\beta}) \right\} \quad (1.77)$$

where  $R$  is the curvature scalar,  $\phi$  is the a scalar field, related via conformal transformations with the dilaton field,  $\chi$ , <sup>7</sup>  $(\nabla\phi)^2$  denotes  $g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ , with  $\nabla_\alpha$  the covariant derivative (repeated indexes means sum over them).  $V(\phi)$  is the quintessence self-interaction potential,  $\chi(\phi)^{-2}$  is the coupling function,  $\mathcal{L}_m$  is the matter Lagrangian,  $\mu$  is a collective name for the matter degrees of freedom. The energy-momentum tensor of background

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<sup>6</sup>Of course with different potentials and varying-mass functions (or, equivalently, coupling functions) than the previous mentioned.

<sup>7</sup>For a discussion about the regularity of the conformal transformation, or the equivalence issue of the two frames, see for example [228, 229, 230, 231, 232, 233, 234, 235, 236, 237] and references therein.

matter is defined by

$$T_{\alpha\beta} = -\frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\alpha\beta}} \left\{ \sqrt{|g|} \chi^{-2} \mathcal{L}(\mu, \nabla\mu, \chi^{-1} g_{\alpha\beta}) \right\}. \quad (1.78)$$

By considering the conformal transformation  $\bar{g}_{\alpha\beta} = \chi(\phi)^{-1} g_{\alpha\beta}$  and defining the Brans-Dicke coupling 'constant'  $\omega(\chi)$  in such way that  $d\phi = \pm \sqrt{\omega(\chi) + 3/2} \chi^{-1} d\chi$  and recalling  $\bar{V}(\chi) = \chi^2 V(\phi(\chi))$  the action (1.77) can be written in the Jordan frame (JF) as (see [2]):

$$S_{JF} = \int_{M_4} d^4x \sqrt{|\bar{g}|} \left\{ \frac{1}{2} \chi \bar{R} - \frac{1}{2} \frac{\omega(\chi)}{\chi} (\bar{\nabla}\chi)^2 - \bar{V}(\chi) + \mathcal{L}_m(\mu, \nabla\mu, \bar{g}_{\alpha\beta}) \right\}, \quad (1.79)$$

where a bar is used to denote geometrical objects defined with respect to the metric  $\bar{g}_{\alpha\beta}$ .

In the STT given by (1.79), the energy-momentum of the matter fields is separately conserved. That is

$$\bar{\nabla}^\alpha \bar{T}_{\alpha\beta} = 0,$$

where

$$\bar{T}_{\alpha\beta} = -\frac{2}{\sqrt{|\bar{g}|}} \frac{\delta}{\delta \bar{g}^{\alpha\beta}} \left\{ \sqrt{|\bar{g}|} \mathcal{L}(\mu, \nabla\mu, \bar{g}_{\alpha\beta}) \right\}.$$

However, when written in de EF (1.77), this is no longer the case (although the overall energy density is conserved). In fact in the EF we find that

$$Q_\beta \equiv \nabla^\alpha T_{\alpha\beta} = -\frac{1}{2} T \frac{1}{\chi(\phi)} \frac{d\chi(\phi)}{d\phi} \nabla_\beta \phi, \quad T = T^\alpha_\alpha.$$

By making use of the above 'formal' conformal equivalence between the Einstein and Jordan frame we can find, for example, that the theory formulated in the EF with the coupling function  $\chi(\phi) = \chi_0 \exp((\phi - \phi_0)/\varpi)$ ,  $\varpi \equiv \pm \sqrt{\omega_0 + 3/2}$  and potential  $V(\phi) = \beta \exp((\alpha - 2)\varpi/(\phi - \phi_0))$  corresponds to the Brans-Dicke theory (BDT) with a powerlaw potential, i.e.,  $\omega(\chi) = \omega_0$ ,  $\bar{V}(\chi) = \beta \chi^\alpha$ . Exact solutions with exponential couplings and exponential potentials (in the EF) were investigated in [200].

It was found (see [2] and references therein) that typically at early times ( $t \rightarrow 0$ ) the BDT solutions are approximated by the vacuum solutions and at late times ( $t \rightarrow \infty$ ) by matter dominated solutions, in which the matter is dominated by the BD scalar field (denoted by  $\chi$  in the Jordan frame). Exact perfect fluid solutions in STT of gravity with a non-constant BD parameter  $\omega(\chi)$  have been obtained by various authors (see [238]).

### $f(R)$ Cosmology

In the metric formalism the action for  $f(R)$ -gravity is given by [239, 240]

$$\mathcal{S}_{\text{metric}} = \int d^4x \sqrt{-g} [f(R) - 2\Lambda + \mathcal{L}^{(m)}], \quad (1.80)$$

where  $f(R)$  is a function of the Ricci scalar  $R$ , and  $\mathcal{L}^{(m)}$  accounts for the matter content of the universe. Additionally, we use the metric signature  $(-1, 1, 1, 1)$ , Greek indices run from 0 to 3, and we impose the standard units in which  $c = 8\pi G = 1$ . Finally, in the following, and without loss of generality, we set the usual cosmological constant  $\Lambda = 0$ .

The fourth-order equations obtained by varying the action (1.80) with respect to the metric write:

$$G_{\mu\nu} = \frac{T_{\mu\nu}^{(m)}}{f'(R)} + T_{\mu\nu}^R, \quad (1.81)$$

where the prime denotes differentiation with respect to  $R$ . In this expression  $T_{\mu\nu}^{(m)}$  denotes the matter energy-momentum tensor, which is assumed to correspond to a perfect fluid with energy density  $\rho_M$  and pressure  $p_m$ . Additionally,  $T_{\mu\nu}^R$  denotes a correction term describing a “curvature-fluid” energy-momentum tensor of geometric origin [119, 120]:

$$T_{\mu\nu}^R = \frac{1}{f'(R)} \left[ \frac{1}{2} g_{\mu\nu} (f(R) - Rf'(R)) + \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \square f'(R) \right], \quad (1.82)$$

where  $\nabla_\mu$  is the covariant derivative associated to the Levi-Civita connection of the metric and  $\square \equiv \nabla^\mu \nabla_\mu$ . Note that in the last two terms of the right hand side there appear fourth-order metric-derivatives, justifying the name “fourth order gravity” used for this class of theories [240]. By taking the trace of equation (1.81) and re-ordering terms one obtains the “trace equation” (equation (5) in section IIA of [84])

$$3\square f'(R) + Rf'(R) - 2f(R) = T, \quad (1.83)$$

where  $T = T^\mu_\mu$  is the trace of the energy-momentum tensor of ordinary matter.

In the phenomenological fluid description of a general matter source, the standard decomposition of the energy-momentum tensor  $T_{\mu\nu}$  with respect to a timelike vector field  $u^\mu$  is given by

$$T_{\mu\nu} = \mu u_\mu u_\nu + 2q_{(\mu} u_{\nu)} + Ph_{\mu\nu} + \pi_{\mu\nu}, \quad (1.84)$$

where  $\mu$  denotes the energy density scalar,  $P$  is the isotropic pressure scalar,  $q_\mu$  is the energy current density vector ( $q_\mu u^\mu = 0$ ) and  $\pi_{\mu\nu}$  is the trace-free anisotropic pressure tensor ( $\pi_{\mu\nu} u^\nu = 0$ ,  $\pi^\mu_\mu = 0$ ,  $\pi_{\mu\nu} = \pi_{\nu\mu}$ ).

The matter fields need to be related through an appropriate thermodynamical equation of state in order to provide a coherent picture of the physics underlying the fluid space-time scenario. Applying this covariant decomposition to the “curvature-fluid” energy-

momentum tensor (1.82) we obtain

$$\begin{aligned}\mu &= -\frac{1}{2} \left[ \frac{f(R) - Rf'(R) + 6H \frac{d}{dt} f'(R)}{f'(R)} \right] \\ P &= -\frac{1}{2} \left[ \frac{-f(R) + Rf'(R) - 4H \frac{d}{dt} f'(R) - 2 \frac{d^2}{dt^2} f'(R)}{f'(R)} \right].\end{aligned}\quad (1.85)$$

Finally, the anisotropic pressure tensor is given by  $\pi_\nu^\mu = \text{diag}(0, -2\pi_+, \pi_+, \pi_+)$ , where

$$\pi_+ = -\frac{\frac{d}{dt} f'(R)}{f'(R)} \sigma_+. \quad (1.86)$$

For HOG theories derived from Lagrangians of the form

$$L = \frac{1}{2} F(R) \sqrt{-g} + \mathcal{L}_{\text{matter}}(\mu, \nabla \mu, g_{\alpha\beta}), \quad (1.87)$$

it is well known that under the conformal transformation,  $\tilde{g}_{\mu\nu} = F'(R) g_{\mu\nu}$ , the field equations reduce to the Einstein field equations with a scalar field  $\phi$  as an additional matter source, where

$$\phi = \sqrt{\frac{3}{2}} \ln F'(R). \quad (1.88)$$

Assuming that (1.88) can be solved for  $R$  to obtain a function  $R(\phi)$ , the potential of the scalar field is given by

$$V(R(\phi)) = \frac{1}{2(F')^2} (RF' - f), \quad (1.89)$$

and quadratic gravity,  $F(R) = R + \alpha R^2$  with the potential

$$V(\phi) = \frac{1}{8\alpha} \left( 1 - e^{-\sqrt{2/3}\phi} \right)^2$$

is a typical example. The restrictions on the potential in the papers [241, 242, 243, 244] were used in [245] to impose conditions on the function  $f(R)$  with corresponding potential (1.89). The conformal equivalence can be formally obtained by conformally transforming the Lagrangian (1.87) and the resulting action becomes [246],

$$\begin{aligned}\tilde{S} &= \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} \tilde{R} - (\nabla \phi)^2 - V(\phi) + \right. \\ &\quad \left. e^{-2\sqrt{2/3}\phi} \mathcal{L}_m(\mu, \nabla \mu, e^{-\sqrt{2/3}\phi} \tilde{g}) \right\}.\end{aligned}\quad (1.90)$$

It is easy to note that the model arising from the action (1.80) can be obtained from (1.77) with the choice  $\chi(\phi) = e^{\sqrt{2/3}\phi}$ .



Recently, a power-counting renormalizable, ultra-violet (UV) complete theory of gravity was proposed by Hořava in [132, 133, 134, 135]. Although presenting an infrared (IR) fixed point, namely General Relativity, in the UV the theory possesses a fixed point with an anisotropic, Lifshitz scaling between time and space of the form  $\mathbf{x} \rightarrow \ell \mathbf{x}$ ,  $t \rightarrow \ell^z t$ , where  $\ell$ ,  $z$ ,  $\mathbf{x}$  and  $t$  are the scaling factor, dynamical critical exponent, spatial coordinates and temporal coordinate, respectively.

Due to these novel features, there has been a large amount of effort in examining and extending the properties of the theory itself [247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263]. Additionally, application of Hořava-Lifshitz gravity as a cosmological framework gives rise to Hořava-Lifshitz cosmology, which proves to lead to interesting behavior [264, 265]. In particular, one can examine specific solution subclasses [266, 267, 268, 269, 270, 271], the perturbation spectrum [272, 273, 274, 275, 276, 277, 278], the gravitational wave production [279, 280, 281], the matter bounce [282, 283, 284], the black hole properties [285, 286, 287, 288, 289, 290, 291, 292, 293, 294], the dark energy phenomenology [295, 296, 297, 298], the astrophysical phenomenology [299, 300] etc. However, despite this extended research, there are still many ambiguities if Hořava-Lifshitz gravity is reliable and capable of a successful description of the gravitational background of our world, as well as of the cosmological behavior of the universe [263, 253, 254].

We begin with a brief review of Hořava-Lifshitz gravity. The dynamical variables are the lapse and shift functions,  $N$  and  $N_i$  respectively, and the spatial metric  $g_{ij}$  (roman letters indicate spatial indices). In terms of these fields the full metric is

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (1.91)$$

where indices are raised and lowered using  $g_{ij}$ . The scaling transformation of the coordinates reads ( $z=3$ ):

$$t \rightarrow l^3 t \quad \text{and} \quad x^i \rightarrow l x^i. \quad (1.92)$$

Decomposing the gravitational action into a kinetic and a potential part as  $S_g = \int dt d^3x \sqrt{g} N (\mathcal{L}_K + \mathcal{L}_V)$ , and under the assumption of detailed balance [134] (the extension beyond detail balance will be performed later on), which apart from reducing the possible terms in the Lagrangian it allows for a quantum inheritance principle [132] (the  $D + 1$  dimensional theory acquires the renormalization properties of the  $D$ -dimensional

one), the full action of Hořava-Lifshitz gravity is given by

$$S_g = \int dt d^3x \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4} C_{ij} C^{ij} + \frac{\kappa^2 \mu}{2w^2} \frac{\epsilon^{ijk}}{\sqrt{g}} R_{il} \nabla_j R_k^l - \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} + \frac{\kappa^2 \mu^2}{8(1-3\lambda)} \left[ \frac{1-4\lambda}{4} R^2 + \Lambda R - 3\Lambda^2 \right] \right\}, \quad (1.93)$$

where

$$K_{ij} = \frac{1}{2N} (g_{ij} - \nabla_i N_j - \nabla_j N_i), \quad (1.94)$$

is the extrinsic curvature and

$$C^{ij} = \frac{\epsilon^{ijk}}{\sqrt{g}} \nabla_k \left( R_i^j - \frac{1}{4} R \delta_i^j \right) \quad (1.95)$$

the Cotton tensor, and the covariant derivatives are defined with respect to the spatial metric  $g_{ij}$ .  $\epsilon^{ijk}$  is the totally antisymmetric unit tensor,  $\lambda$  is a dimensionless constant and  $\Lambda$  is a negative constant which is related to the cosmological constant in the IR limit. Finally, the variables  $\kappa$ ,  $w$  and  $\mu$  are constants with mass dimensions  $-1$ ,  $0$  and  $1$ , respectively.

In order to add the dark-matter content in a universe governed by Hořava gravity, a scalar field is introduced [264, 265], with action:

$$S_M \equiv S_\phi = \int dt d^3x \sqrt{g} N \left[ \frac{3\lambda-1}{4} \frac{\dot{\phi}^2}{N^2} + m_1 m_2 \phi \nabla^2 \phi - \frac{1}{2} m_2^2 \phi \nabla^4 \phi + \frac{1}{2} m_3^2 \phi \nabla^6 \phi - V(\phi) \right], \quad (1.96)$$

where  $V(\phi)$  acts as a potential term and  $m_i$  are constants. Although one could just follow a hydrodynamical approximation and introduce straightaway the density and pressure of a matter fluid [253], the field approach is more robust, especially if one desires to perform a phase-space analysis.

Now, in order to focus on cosmological frameworks, we impose the so called projectability condition [263] and use an FRW metric,

$$N = 1, \quad g_{ij} = a^2(t) \gamma_{ij}, \quad N^i = 0, \quad (1.97)$$

with

$$\gamma_{ij} dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2, \quad (1.98)$$

where  $k = -1, 0, 1$  correspond to open, flat, and closed universe respectively. In addition, we assume that the scalar field is homogenous, i.e  $\phi \equiv \phi(t)$ .

#### Ad hoc modifications of the Friedmann equation

Even though the most popular explanation to the late-time acceleration in the universe is the existence of some kind of dark energy (perhaps a scalar field), this is not the only possibility. Recently, Freese and Lewis [41] proposed the so-called Cardassian models as an alternative explanation which involves only matter and radiation and does not invoke either vacuum energy or a cosmological constant. In these models the universe has a flat geometry, as required by measurements of the cosmic background radiation [301] and it is filled only with radiation and matter (baryonic or not). The Friedman equation is modified with respect to its usual form by the addition of a term in its right hand side, specifically

$$3H^2 = \rho + \sigma\rho^n, \quad (1.99)$$

in units such that  $8\pi/m_{pl}^2 = 1$  and with  $\sigma > 0$  being an arbitrary constant.

For  $n < 1$  the second term becomes important if  $z < \mathcal{O}(1)$ . From there on it dominates the Friedmann equation and yields  $a \propto t^{2/3n}$  for ordinary matter, so there will be acceleration provided  $n < 2/3$ . There are two main (possibly unrelated) motivations for the  $\rho^n$  modifications: (1) As shown in [302], terms of that form typically appear in the Friedman equation when the universe is embedded as a three-dimensional surface (3-brane) in higher dimensions. (2) Alternatively, these functions may appear in a purely four dimensional theory in which the modified right hand side of the Friedman equation is due to an extra contribution to the total energy density. One will then regard the right hand side of the Friedman equation as corresponding to a single fluid, with an extra contribution to the energy-density tensor in the (ordinary four dimensional) Einstein equations.

The interpretation of the Cardassian expansion as due to an interacting dark matter fluid with negative pressure was developed in [303]. The Cardassian term on the right hand side of the Friedman equation is interpreted as the interacting term and gives rise to the effective negative pressure which drives the cosmological acceleration.

Interestingly, Cardassian models survive several observational tests, the most significant being that it allows for a universe consisting of just matter and radiation. The energy density giving a closed universe  $\rho_c$  is much smaller than its counterpart standard cosmology  $\rho_{c,old}$  (specifically  $\rho_c = \rho_{c,old} [1 + (1 + z_{eq})^3(1 - n)]^{-1}$ ), and matter alone is enough to provide a flat geometry.

To illustrate how the accelerated expansion is possible in such framework, let us assume for sake that the matter content is dust, i.e.,  $\rho \propto a^{-3} \propto (1 + z)^3$ . With this hypothesis, the correction term dominates for redshifts  $z < z_{eq}$  with  $z_{eq}$  defined by

$$(1 + z_{eq})^{3(1-n)} = \frac{\sigma\rho_0}{3H^2 - \sigma\rho_0}, \quad (1.100)$$

( $\rho_0$  is the current energy density). Once the correction term dominates, the scale factor and its derivatives up to second order are given respectively by:

$$a \propto t^{\frac{2}{3n}}, \quad \dot{a} \propto \frac{2}{3n} t^{-1+\frac{2}{3n}}, \quad \ddot{a} \propto \frac{2(2-3n)}{9n^2} t^{-2(1-\frac{1}{3n})}.$$

Hence, we obtain accelerated expansion provided  $n < 2/3$ . Therefore, Cardassian models explains very well the current accelerated expansion only considering ordinary matter and radiation.

#### Crossing the “phantom divide”

The quintom paradigm is a hybrid construction of a quintessence component, usually modelled by a real scalar field that is minimally coupled to gravity, and a phantom field: a real scalar field –minimally coupled to gravity– with negative kinetic energy. Let us define the equation of state parameter of any cosmological fluid as  $w \equiv \text{pressure/density}$ . The simplest model of dark energy (vacuum energy or cosmological constant) is assumed to have  $w = -1$ . A key feature of quintom-like behavior is the crossing of the so called phantom divide, in which the equation of state parameter crosses through the value  $w = -1$ .

In [304] uncorrelated and nearly model independent band power estimates (basing on the principal component analysis [305]) of the EoS of DE and its density as a function of redshift were presented, by fitting to the SNIa data. Quite unexpectedly, they found marginal ( $2\sigma$ ) evidence for  $w(z) < -1$  at  $z < 0.2$ , which is consistent with other results in the literature [26, 164, 306, 307, 308, 309].

The aforementioned result implied that the EoS of DE could indeed vary with time. Therefore, one could use a suitable parametrization of  $w_{\text{DE}}$  as a function of the redshift  $z$ , in order to satisfactory describe such a behavior. There are two well-studied parametrizations. The first (ansatz A) is:

$$w_{\text{DE}} = w_0 + w'z, \quad (1.101)$$

where  $w_0$  the DE EoS at present and  $w'$  an additional parameter. However, this parametrization is only valid at low redshift, since it suffers from severe divergences at high ones, for example at the last scattering surface  $z \sim 1100$ . Therefore, a new, divergent-free ansatz (ansatz B) was proposed [310, 311]:

$$w_{\text{DE}} = w_0 + w_1(1 - a) = w_0 + w_1 \frac{z}{1+z}, \quad (1.102)$$

where  $a$  is the scale factor and  $w_1 = -dw/da$ . This parametrization exhibits a very good behavior at high redshifts.

In [165] the authors used the “gold” sample of 157 SNIa, the low limit of cosmic ages and the HST prior, as well as the uniform weak prior on  $\Omega_m h^2$ , to constrain the free parameters of above two DE parameterizations. As can be seen in Fig.1.1 they found that

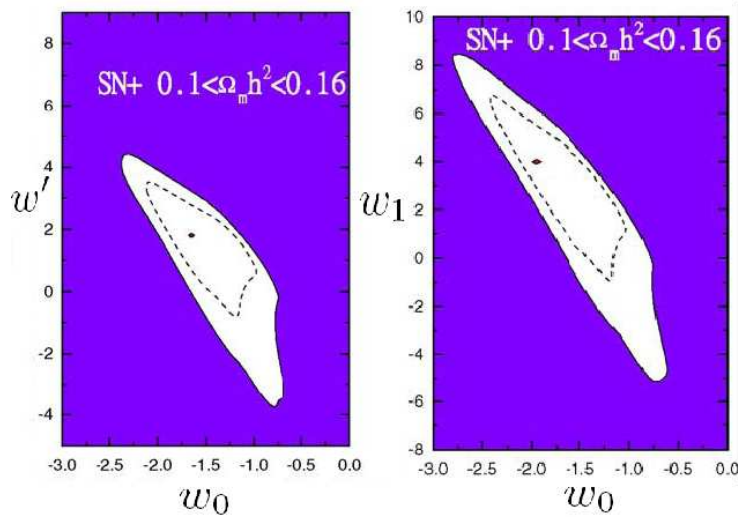


Figure 1.1: Two-dimensional contour-plots of the DE equation-of-state parameters, in two different parameterizations and using SNIa data. The left graph corresponds to ansatz A (expression (1.101)) and the right graph are to ansatz B (expression (1.102)). From Ref. [165].

the data seem to favor an evolving DE with the EoS being below  $-1$  around the present epoch, while it was in the range  $w > -1$  in the near cosmological past. This result holds for both parametrizations (1.101),(1.102), and in particular the best fit value of the EoS at present is  $w_0 < -1$ , while its “running” coefficient is larger than 0.

Apart from the SNIa data, CMB and LSS data can be also used to study the variation of EoS of DE. In [312], the authors used the first year WMAP, SDSS and 2dFGRS data to constrain different DE models. They indeed found that evidently the data favor a strongly time-dependent  $w_{DE}$ , and this result is consistent with similar project of the literature [313, 314, 315, 316, 317, 318, 319, 320, 321, 322]. Using the latest 5-year WMAP data, combined with SNIa and BAO data, the constraints on the DE parameters of ansatz B are:  $w_0 = -1.06 \pm 0.14$  and  $w_1 = 0.36 \pm 0.62$  [323, 324, 325], and the corresponding contour plot is presented in Fig.1.2.

In conclusion, as can be observed, the current observational data mildly favor  $w_{DE}$  crossing the phantom divide during the evolution of universe.

Some interesting general aspects of the problem of the phantom divide crossing were discussed in [326], where the viability requirements on the equation of state and sound speed were analyzed. Some of realizations of the crossing can have an extradimensional origin, either in the brane [327, 328, 329, 330], or the string gas context [331]. Other worth mentioning alternatives are models in the framework of scalar-tensor theories [45, 185, 332]; a single field proposal involving high order derivative operators in the lagrangian [333]; and an interacting Chaplygin gas [334]. The impossibility of the occurrence of the transition in traditional single field models (minimally coupled to mat-

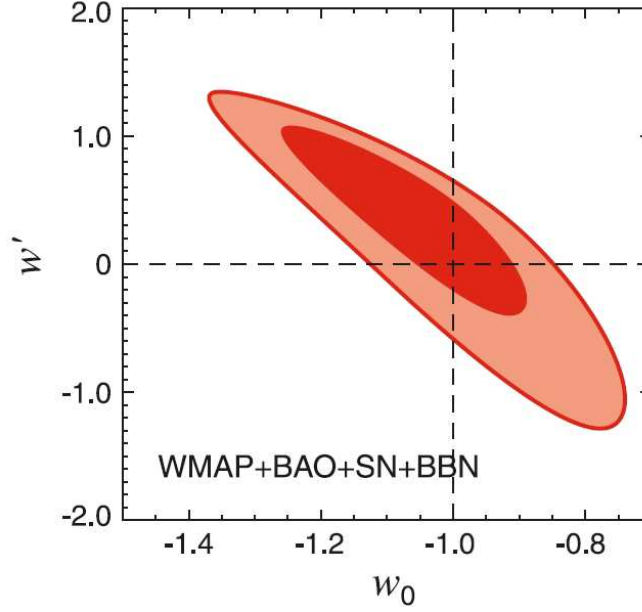


Figure 1.2: Two-dimensional contour-plot of the DE equation-of-state parameters, in parameterization ansatz B (expression (1.102)), and using WMAP, BAO, SNIa data. From Ref. [323].

ter) [43, 44], has motivated much activity in the construction of two field models that do the job. Examples of explicit constructions can be found in [58, 335, 336, 337], but perhaps the class of models which have received most attention are quintom cosmologies [51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69]. Quintom behavior (i.e., the  $w = -1$  crossing) has been investigated in the context of h-essence cosmologies [58, 59]; in the context of holographic dark energy [70, 71, 72, 73, 74]; inspired by string theory [61, 62, 63]; derived from spinor matter [64]; for arbitrary potentials [66, 67, 68, 69]; using isomorphic models consisting of three coupled oscillators, one of which carries negative kinetic energy (particularly for investigating the dynamical behavior of massless quintom)[338]. The crossing of the phantom divide is also possible in the context of scalar tensor theories [46, 47, 48, 49, 50] as well as in modified theories of gravity [75].

The two-field quintom model has Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\phi, \varphi), \quad (1.103)$$

The cosmological evolution of quintom model with exponential potential has been examined, from the dynamical systems viewpoint, in [56] and [57, 60]. The difference between [56] and [57, 60] is that in the second case the potential considers the interaction between the conventional scalar field and the phantom field. In [57] it had been proven that in the absence of interactions, the solution dominated by the phantom field should be

the attractor of the system and the interaction does not affect its attractor behavior. In [60] the case in which the interaction term dominates against the mixed terms of the potential, was studied. It was proven there, that the hypothesis in [57] is correct only in the cases in which the existence of the phantom phase excludes the existence of scaling attractors (in which the energy density of the quintom field and the energy density of DM are proportional). Some of this results were extended in [66], for arbitrary potentials. There it was settled down under what conditions on the potential it is possible to obtain scaling regimes. It was proved there, that for arbitrary potentials having asymptotic exponential behavior, scaling regimes are associated to the limit where the scalar fields diverge. Also it has been proven that the existence of phantom attractors in this framework is not generic and consequently the corresponding cosmological solutions lack the big rip singularity.





# Chapter 2

## Qualitative theory of dynamical systems

In this chapter we briefly review some results of the qualitative theory of dynamical systems, settling the theoretical basis for the qualitative study of concrete cosmological models.

### 2.1 Introduction

The works of H. J. Poincaré in Celestial Mechanics [339, 340] settled the basis for the local and global analysis of non-linear differential equations, particularly, for the stability theory of singular points and periodic orbits, stable and unstable manifolds, etc.

After H. J. Poincaré, and following the studies of J. Hadamard about geodesic fluids [341], G. D. Birkhoff studied the complex structure of orbits arising when a complete integrable system is perturbed [342, 343]. Therefore, the basic question is how prevalent integrability is? The answer was given by A. N. Kolmogorov (1954), V. I. Arnold (1963) and J. K. Moser (1973), in which is now called KAM theorem, the basic theorem on chaos for Hamiltonian systems [344, 345]. Some of the more important theorems of stability were given by A. M. Liapunov [346], offering a method to determine the stability of the singular points when the information obtained by linearization is not conclusive. This theory is a vast area belonging to the theory of dynamical systems [347, 348, 349]. Finally, in the XX's, was possible to formulate a geometric theory of dynamical systems, mainly due to the works of V. I. Arnold [350, 351].

Qualitative methods have been proved to be a powerful scheme for investigating the physical behavior of cosmological models. It has been used three different approaches: approximation by parts, Hamiltonian methods, and dynamical systems methods [1]. In the third case the Einstein's field equations of Bianchi's cosmologies and its isotropic subclass (FLRW models), can be written as an autonomous system of first-order differential equations whose solution curves partitioned to  $\mathbb{R}^n$  in orbits, defining a dynamical system in  $\mathbb{R}^n$ . In the general case, the elements of the phase space partition (i.e., singular points,

invariant sets, etc.) can be listed and described. This study consists of several steps: determining the singular points, the linearization in a neighborhood of them, the search for the eigenvalues of the associated Jacobian matrix, checking the stability conditions in a neighborhood of the singular points, the finding of the stability and instability sets and the determination of the basin of attraction, etc. In some occasions, in order to do that, it is needed to simplify a dynamical system. Two approaches are applied to this objective: one, reduce the dimensionality of the system and two, eliminate the nonlinearity. Two rigorous mathematical techniques that allow substantial progress along both lines are center manifold theory and the method of normal forms. We submit the reader to sections 2.2.5.3 and 2.2.5.4 for a summary about such techniques.

The most general result to be applied in order to determine the asymptotic stability of a singular point,  $a$ , is Lyapunov's stability theorem. Lyapunov's stability method provides information not only about the asymptotic stability of a given singular point but also about its basin of attraction. This cannot be obtained by the usual methods found in the literature, such as linear stability analysis or first-order perturbation techniques. Moreover, Lyapunov's method is also applicable to non-autonomous systems. To our knowledge, there are few works that use Lyapunov's method in cosmology [352, 353, 354, 355, 356]. In [355] it is investigated the general asymptotic behavior of Friedman-Robertson-Walker (FRW) models with an inflaton field, scalar-tensor FRW cosmological models and diagonal Bianchi-IX models by means of Lyapunov's method. In [356] it is investigated the stability of isotropic cosmological solutions for two-field models in the Bianchi I metric. The author proved that the conditions sufficient for the Lyapunov stability in the FRW metric provide the stability with respect to anisotropic perturbations in the Bianchi I metric and with respect to the cold dark matter energy density fluctuations. Sufficient conditions for the Lyapunov's stability of the isotropic singular points of the system of the Einstein equations are also found (these conditions coincided with the previously obtained in [357] for the quintom paradigm without using Liapunov's technique). To apply Lyapunov's stability method it is required the construction of the so-called strict Lyapunov's function, i.e., a  $C^1$  function  $V : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  defined in a neighborhood  $U$  of  $a$  such that  $V(a) = 0, V(x) > 0, x \neq a$  and  $\dot{V}(x) \leq 0 (< 0)$  in  $U \setminus \{a\}$ . The construction of such  $V$  is laborious, and sometimes impossible. One alternative way is to follow point (5) in section 2.3.

For the investigation of hyperbolic singular points of autonomous vector fields we can use Hartman-Grobman's theorem (theorem 19.12.6 en [358] p. 350) which allows for analyzing the stability of an singular point from the linearized system around it. For isolated nonhyperbolic singular points we can use normal forms theorem (theorem 2.3.1 in [359]), which contains Hartman-Grobman's theorem as particular case. The aim of the normal form calculation is to construct a sequence of non-linear transformations which successively remove the non-resonant terms in the vector field of order  $r$  in the Taylor's

expansion,  $\mathbf{X}_r$ , starting from  $r = 2$ . Normal forms (NF) theory has been used in the context of cosmological models in order to get useful information about isolated non-hyperbolic singular points. In [151] was investigated Hořava-Lifshitz cosmology from the dynamical system view point. There was proved the stability of de Sitter solution (corresponding to a non-hyperbolic singular point) in the case where the detailed-balance condition is relaxed, using NF expansions. In [360] were investigated closed isotropic models in second order gravity. There the normal form of the dynamical system has periodic solutions for a large set of initial conditions. This implies that an initially expanding closed isotropic universe may exhibit oscillatory behavior.

The more relevant concepts of qualitative theory are the concept of flow, and the concept of invariant manifold. The invariant manifold theorem (theorem 3.2.1 en [358]), that claims for the existence of local stable and unstable manifolds (under suitable conditions for the vector field), only allows to obtain partial information about the stability of singular points and does not gives a method for determining the stable and unstable manifolds. Sometimes it is required to consider higher order terms in the Taylor's expansion of the vector field (e.g. normal forms). For the investigation of the asymptotic states of the system the appropriated concepts are  $\alpha$  and  $\omega$ -limit sets of  $x \in \mathbb{R}^n$  (definition 8.1.2 en [358] p. 105). To characterize these invariant sets one can use the LaSalle's Invariance Principle ([361]; theorem 8.3.1 [358], p. 111) or the Monotonicity Principle ([1], p. 103; [362] p. 536). To apply the Monotonicity Principle it is required the construction of a monotonic function. In some cases a monotonic function is suggested by the form of the differential equation (see equation (2.50) below) and in some cases by the Hamiltonian formulation of the field equations. In [363] is given a prescription of how to find monotonic functions for Bianchi cosmologies using Hamiltonian techniques, merely as a convenient tool for an intermediate step; the final results are described in terms of scale-automorphism invariant Hubble-normalized reduced state vector, which is independent of a Hamiltonian formulation. Also, one can use the Poincaré-Bendixson ([2] p. 22, theorem 2 [364] p. 6, [348]) theorem and its collorary to distinguishing among all possible  $\omega$ -limit sets in the plane. From the Poincaré-Bendixon Theorem follows that any compact asymptotic set is one of the following; 1.) a singular point, 2.) a periodic orbit, 3.) the union of singular points and heteroclinic or homoclinic orbits. As a consequence if the existence of a closed (i.e., periodic, heteroclinic or homoclinic) orbit can be ruled out it follows that all asymptotic behavior is located at a singular point. To ruled out a closed orbits for two-dimensional systems we can use the Dulac's criterion (theorem 3 [364] p. 6, se also [1], p. 94, and [2]). It requires the construction of a Dulac's function. A Dulac's function,  $B$ , is a  $C^1$  function defined in a simply connected open subset  $D \subseteq \mathbb{R}^2$  such that  $\nabla(Bf) = \frac{\partial}{\partial \sigma_2}(Bf_1) + \frac{\partial}{\partial \sigma_3}(Bf_2) > 0$ , or ( $< 0$ ) for all  $x \in D$  (see [364]).

## 2.2 Qualitative theory of dynamical systems

In this section we discuss some of tools of the qualitative theory of dynamical systems which are applicable, for instance, to the study of cosmological systems.

### 2.2.1 Definitions and basic results

In this book we consider vector fields of the form

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}, \tau; \mu) \quad (2.1)$$

where  $\mathbf{x} \in U \subset \mathbb{R}^n$ , denotes a state vector defined in an open set  $U$  of  $\mathbb{R}^n$ ,  $\tau \in \mathbb{R}$  denotes “time”,  $\mu \in V \subset \mathbb{R}^p$  denotes a vector of parameters defined in an open set  $V$  of  $\mathbb{R}^p$ , and the comma denotes derivative with respect to  $\tau$ . It is assumed that  $\mathbf{X}$  is a function of class  $C^r$ ,  $r \geq 1$  (i.e., a function with continuous partial derivatives of order  $r$  with respect to their arguments) in order to obtain solutions of the same differentiable class. In case that  $\mathbf{X}$  do not depend explicitly of time we say that the vector field is autonomous. In this case, if the parameters are not relevant for the discussion, we write

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}) \quad (2.2)$$

In this case it is assumed that  $\mathbf{X}$  is a function of class  $C^r$ ,  $r \geq 1$  defined in an open set  $U \subset \mathbb{R}^n$ .

A solution of (2.1) is a map,  $\mathbf{x}$ , defined in an interval  $I \subset \mathbb{R}$  taking values in  $\mathbb{R}$  given by

$$\mathbf{x} : I \rightarrow \mathbb{R}, \tau \rightarrow \mathbf{x}(\tau), \quad (2.3)$$

such that it satisfies equation (2.1), i.e.,

$$\mathbf{x}(\tau)' = \mathbf{X}(\mathbf{x}(\tau), \tau; \mu) \quad (2.4)$$

The map (2.3) is interpreted geometrically as a curve in  $\mathbb{R}^n$ , such that  $\mathbf{X}$  in (2.1) represents the tangent vector at each point on the curve. For that reason  $\mathbf{X}$  is referred as a vector field. The space of the dependent variables of (2.1) is referred by phase space of (2.1). In an abstract language, the objective of the qualitative study of a vector field is to obtain information for the understanding of the geometry of the solution curves of (2.1) in the phase space, actually without solving equation (2.1) explicitly.

The solution  $\mathbf{x}(\tau)$  of (2.1), passing through the point  $\mathbf{x} = \mathbf{x}_0$  at time  $\tau = \tau_0$ , is denoted by  $\mathbf{x}(\tau, \tau_0, \mathbf{x}_0; \mu)$ , or  $\mathbf{x}(\tau, \tau_0, \mathbf{x}_0)$ , if the parameters are not relevant for the discussion. To  $\mathbf{x}(\tau, \tau_0, \mathbf{x}_0)$ , it is referred also as trajectory of phase curve passing passing through the

point  $\mathbf{x} = \mathbf{x}_0$  at time  $\tau = \tau_0$ . To the graph

$$\{(\mathbf{x}, \tau) \in \mathbb{R}^n \times \mathbb{R} \mid \mathbf{x} = \mathbf{x}(\tau, \tau_0, \mathbf{x}_0), \tau \in I\},$$

it is referred as integral curve.

Given  $\mathbf{x}_0 \in U \subset \mathbb{R}^n$  located at the phase space of (2.1), the orbit passing through the point  $\mathbf{x} = \mathbf{x}_0$  it is denoted and defined as

$$O(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{x}(\tau, \tau_0, \mathbf{x}_0), \tau \in I\}. \quad (2.5)$$

It is a fact that for all  $T \in I$ ,  $O(\mathbf{x}(T, \tau_0, \mathbf{x}_0)) = O(\mathbf{x}_0)$ .

The positive (future) orbit passing through the point  $\mathbf{x} = \mathbf{x}_0$  it is denoted and defined as

$$O^+(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{x}(\tau, \tau_0, \mathbf{x}_0), \tau \geq \tau_0\}. \quad (2.6)$$

Similarly it is defined the negative (past) orbit passing through the point  $\mathbf{x} = \mathbf{x}_0$ , denoted by  $O^-(\mathbf{x}_0)$ , changing  $\tau \geq \tau_0$  by  $\tau \leq \tau_0$  in (2.6).

In much of the applications the topological structure of the phase space can be more general than  $\mathbb{R}^n$ ; common examples are cylindrical, spherical and toroidal phase spaces. The natural structure to be considered, whenever phase space topology is concerned, is that of topological (differentiable) manifold.

**Definition 1 ((Topological) manifold, [365] pp. 3-4)** *Let  $M$  be a Hausdorff space provided with a numerable basis. Let  $p \in M$ . If there exists a positive number  $m$  (possibly depending on  $p$ ), a neighborhood  $V(p)$  of  $p$  and an homeomorphism  $h : V(p) \rightarrow \mathbb{R}^m$  such that  $h(V(p)) \subset \mathbb{R}^m$  is an open set of  $\mathbb{R}^m$ , then,  $M$  is a (topological) manifold.*

The positive integer  $m$  is unique and it is referred as dimension of  $M$ . To the set  $\{(V_i, h_i)\}$  it is referred as the collection of local coordinates (local charts) of  $M$ .

**Definition 2 (Euclidean semi-space, [365] pp. 3-4)** *The  $n$ -dimensional Euclidean semi-space is denoted and defined by  $\mathbb{H}^n := \{\mathbf{x} \in \mathbb{R}^n : x_n \geq 0\}$*

**Definition 3 ((Topological) manifold with boundary, [365] pp. 3-4)** *Let  $M$  be a Hausdorff space provided with a numerable basis. Let  $p \in M$ . If there exists a positive number  $m$  (possibly depending on  $p$ ), a neighborhood  $V(p)$  of  $p$  and an homeomorphism  $h : V(p) \rightarrow \mathbb{H}^m$  such that  $h(V(p)) \subset \mathbb{H}^m$  is an open set of  $\mathbb{H}^m$ , then,  $M$  is a (topological) manifold with boundary.*

**Definition 4 (Boundary, [365] pp. 3-4)** *Let  $M$  be a manifold with boundary. The boundary of  $M$ ,  $\partial M$ , is defined by  $\partial M := \{p \in M : h(p) \in \mathbb{R}^{m-1} \times \{0\}\}$ .*

This means that  $\partial M$  is a topological manifold of dimension  $m - 1$  consistent of those points  $p \in M$  which are transformed by a chart  $(V, h)$  (and so, by all the charts) in a neighborhood of  $p$  in a point with zero  $m$ -th coordinate, i.e.,  $x_m = 0$ .

**Definition 5 (Interior, [365] pp. 3-4)** *Let  $M$  be a manifold with boundary. The interior of  $M$ ,  $\text{Int}M$ , is defined by  $\text{Int}M := M \setminus \partial M$ .*

**Notes.** It is a fact that  $\mathbb{R}^{m-1} \subset \mathbb{H}^m \subset \mathbb{R}^m$ . Let  $M$  and  $N$  topological manifolds with dimension  $n$  and  $m$  respectively. Then,  $M \times N$  is a topological manifold (with or without boundary) of dimension  $m + n$  and  $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$ .

**Definition 6 (Differentiable manifold, [365] p. 8)**  *$M$  is a differentiable manifold of class  $C^r$  if the local charts  $\{(V_i, h_i)\}$  satisfy:*

1.  $V_i$  is a covering of  $M$ , i.e.,  $M \subset \bigcup V_i$ .
2. If  $(V_1, h_1)$  and  $(V_2, h_2)$  are local charts with  $V_1 \cap V_2 = \emptyset$ , then there exists a local change of charts

$$h_1 \circ (h_2)^{-1} : h_2(V_1 \cap V_2) \rightarrow \mathbb{R}^m$$

which is of differentiable class  $C^r$ .

3. The collection  $\{(V_i, h_i)\}$  is maximal with respect property 2. This means that, if we include a new (different) local chart  $(V_k, h_k)$  to the collection, then, the local change of charts  $h_j \circ (h_k)^{-1}$  (the index  $j$  is referred to a local chart in the collection) is of differentiable class  $C^s$ ,  $s < r$ .

In order to guarantee the existence of solution of (2.1) it is assumed that  $\mathbf{X}(\mathbf{x}, \tau)$  is a function of class  $C^r$ , ( $r \geq 1$ ) defined in an open set  $U \subset \mathbb{R}^n \times \mathbb{R}$ , resulting the

**Theorem 1 (Existence and uniqueness, theorem 7.1.1, [358])** *Let  $(\mathbf{x}_0, \tau_0) \in U$ . Then, there exists a solution of (2.1) passing through the point  $\mathbf{x} = \mathbf{x}_0$  at time  $\tau = \tau_0$ , denoted by  $\mathbf{x}(\tau, \tau_0, \mathbf{x}_0)$ , with  $\mathbf{x}(\tau_0, \tau_0, \mathbf{x}_0) = \mathbf{x}_0$ , for  $|\tau - \tau_0|$  small enough. This solution is unique in the sense that for any other solution passing through the point  $\mathbf{x} = \mathbf{x}_0$  at time  $\tau = \tau_0$ , is the same than  $\mathbf{x}(\tau, \tau_0, \mathbf{x}_0)$ , in the common existence interval. Besides  $\mathbf{x}(\tau, \tau_0, \mathbf{x}_0)$  is a  $C^r$ , ( $r \geq 1$ ) function of  $\tau, \tau_0$  and  $\mathbf{x}_0$ .*

**Proof.** See [349, 348, 366]

Theorem 1 only guarantees existence and uniqueness for infinitesimal time intervals. This theorem can be reapplied to extend univocally the time interval of existence as expressed in the following theorem.

Let  $C \subset U \subset \mathbb{R}^n \times \mathbb{R}$  a compact set containing to  $(\mathbf{x}_0, \tau_0)$ .

**Theorem 2 (Extendibility, theorem 7.2.1, [358])** *The solution  $\mathbf{x}(\tau, \tau_0, \mathbf{x}_0)$ , can be extended backward and forward in time  $\tau$  to the boundary of  $C$ .*

**Proof.** See [366].

The next theorem for autonomous vector fields, shows that if  $\|\mathbf{X}(\mathbf{x})\|$  does not increase too rapidly as  $\|\mathbf{x}\| \rightarrow +\infty$  then all solutions can be extended indefinitely.

**Theorem 3 (Extendibility, theorem 4.3, [1], p. 87)** *If  $\mathbf{X}$  is continuous, then there exists a constant  $M$  such that  $\|\mathbf{X}(\mathbf{x})\| \leq M\|\mathbf{x}\|$  for all  $\mathbf{x} \in \mathbb{R}^n$ , then any solution of 2.2 is defined for all  $\tau \in \mathbb{R}$ .*

Theorem 3 implies that one can modify a given autonomous vector field 2.2, so that the orbits are unchanged, but such that all solutions are defined for all  $\tau \in \mathbb{R}$ . The idea is to re-scale the vector field  $\mathbf{X}$  so as to make it bounded,

$$\mathbf{X}(\mathbf{x}) \rightarrow \lambda(\mathbf{x})\mathbf{X}(\mathbf{x}),$$

where  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$  is a  $C^1$  function which is positive on  $\mathbb{R}^n$ , in order to preserve the time direction of time (e.g.,  $\lambda(\mathbf{x}) = (1 + \|\mathbf{X}(\mathbf{x})\|)^{-1}$  will suffice).

**Theorem 4 (Corollary, 4.2, [1], p. 87)** *If  $\mathbf{X}$  is  $C^1$ , and  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $C^1$  and positive, then 2.2 and  $\mathbf{x}' = \lambda(\mathbf{x})\mathbf{X}(\mathbf{x})$  have the same orbits, and  $\lambda$  can be chosen so that all solutions of 2.2 are defined for all  $\tau \in \mathbb{R}$ .*

Given the vector field (2.1), such that  $\mathbf{X}(\mathbf{x}, \tau; \mu)$  is of class  $C^r$ , ( $r \geq 1$ ) defined in an open set  $U \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^p$ , then it is verified

**Theorem 5 (Differentiability solutions, theorem 7.3.1, [358])** *Let  $(\mathbf{x}_0, \tau_0, \mu) \in U$ . Then, the solution  $\mathbf{x}(\tau, \tau_0, \mathbf{x}_0; \mu)$  is a  $C^r$  ( $r \geq 1$ ) function of  $\tau, \tau_0, \mathbf{x}_0$  and  $\mu$ .*

**Proof.** See [349, 366].

## 2.2.2 Desirable stability properties of nonlinear vector fields

Let be the vector field (2.1) such that  $\mathbf{X}(\mathbf{x}, \tau)$  is a function at least continuous in  $\tau$  and of class  $C^r$ , ( $r \geq 2$ ) with respect to the components of  $\mathbf{x}$ . Let  $\mathbf{x} = \bar{\mathbf{x}}(\tau)$  any solution of (2.1).

Roughly speaking,  $\bar{\mathbf{x}}(\tau)$  is stable if the solutions, close to  $\bar{\mathbf{x}}(\tau)$  at some given initial time, remain close to  $\bar{\mathbf{x}}(\tau)$  all future time. The solution is asymptotically stable if the near solutions not only remain close, but tend to  $\bar{\mathbf{x}}(\tau)$  as time goes forward.

Let us formalize these ideas. Given the solution  $\mathbf{x} = \bar{\mathbf{x}}(\tau)$  of (2.1) defined for  $\tau_0 \leq \tau < +\infty$ , then the deviations of the vector field with respect to  $\bar{\mathbf{x}}(\tau)$  is given by the variable  $\mathbf{y} = \mathbf{x} - \bar{\mathbf{x}}(\tau)$  with vector field

$$\mathbf{y}' = \mathbf{Y}(\mathbf{y}, \tau), \quad (2.7)$$

where  $\mathbf{Y}(\mathbf{y}, \tau) = \mathbf{X}(\mathbf{y} + \bar{\mathbf{x}}(\tau), \tau) - \mathbf{X}(\bar{\mathbf{x}}(\tau), \tau)$ . Using this transformation of variables then  $\mathbf{y} = \mathbf{0}$  is the stationary solution of the vector field (2.7). Hence the stability analysis of the solution  $\bar{\mathbf{x}}(\tau)$  of (2.1) is reduce to the study of the stability of the stationary solution  $\mathbf{y} = \mathbf{0}$  of the associated problem (2.7).

For convenience we consider the original notation assuming that  $\mathbf{X}(\mathbf{0}, \tau) = \mathbf{0}$  for  $\tau_0 \leq \tau < +\infty$ .

Now we enumerate four basic definitions of stability that are desirable properties of a nonlinear vector field [367]:

1. The solution  $\mathbf{x} = \mathbf{0}$  is said to be stable if given any tolerance  $\epsilon > 0$  and any initial time  $\tau_0$ , there exists a restriction  $\delta = \delta(\epsilon, \tau_0) > 0$  such that  $\|\mathbf{x}_0\| < \delta$  implies that  $\mathbf{x}(\tau, \tau_0, \mathbf{x}_0)$  exists for  $\tau_0 \leq \tau < +\infty$  and satisfies  $\|\mathbf{x}(\tau, \tau_0, \mathbf{x}_0)\| < \epsilon$  for all  $\tau \geq \tau_0$ . Therefore, any solution that stars close to  $\mathbf{x} = \mathbf{0}$  will remains close to it all future time.
2. The solution  $\mathbf{x} = \mathbf{0}$  is said to be asymptotically stable if it is stable and there exists a restriction  $\delta_1 = \delta_1(\epsilon, \tau_0) > 0$  such that  $\|\mathbf{x}_0\| < \delta_1$  implies that  $\mathbf{x}(\tau, \tau_0, \mathbf{x}_0) \rightarrow \mathbf{0}$  when  $\tau \rightarrow +\infty$ . Therefore,  $\mathbf{x} = \mathbf{0}$  is stable and any solution that stars close to the origin tends to it as time goes forward.
3. The solution  $\mathbf{x} = \mathbf{0}$  is said to be uniformly asymptotically stable if it is stable with restriction  $\delta$  independently of  $\tau_0$  (i.e.,  $\delta = \delta(\epsilon) > 0$ ) and for any given tolerance  $\epsilon > 0$ , there exists a number  $T = T(\epsilon)$  with the property

$$\lim_{\epsilon \rightarrow 0} T(\epsilon) = +\infty$$

such that  $\tau - \tau_0 > T(\epsilon)$  implies  $\|\mathbf{x}(\tau, \tau_0, \mathbf{x}_0)\| < \epsilon$ . Therefore,  $\mathbf{x}$  tends to zero as  $\tau - \tau_0 \rightarrow +\infty$ , uniformly in  $\tau_0$  and in  $\mathbf{x}_0$ .

4. The solution  $\mathbf{x} = \mathbf{0}$  is said to be exponentially asymptotically stable if there exist positive constants  $\delta, K$  and  $\alpha$ , such that  $\|\mathbf{x}_0\| < \delta$  implies

$$\|\mathbf{x}(\tau, \tau_0, \mathbf{x}_0)\| \leq K e^{-\alpha(\tau - \tau_0)} \|\mathbf{x}_0\|,$$

for all  $\tau \geq \tau_0$ .



The stability definitions 1-4 are used in many contexts. Stability and asymptotic stability are not robust properties in the sense that they are not preserved if the system is influenced by small perturbations of the vector field. However, uniformly asymptotic stability and exponential asymptotic stability are robust in the previous sense. For autonomous systems, asymptotic stability and uniform asymptotic stability are equivalent. A solution which is exponentially asymptotically stable is also uniformly asymptotically stable.

The above definitions describe mathematically different types of stability; however, they do not provide a method for determine when a given solution is stable or not. Now let us concentrate our attention to this question

### 2.2.3 Linearization

In order to determine the stability of  $\bar{\mathbf{x}}(\tau)$  we need to understand the nature of the solutions nearby to  $\bar{\mathbf{x}}(\tau)$ . In order to do so, it is natural to define the vector  $\mathbf{y}$  by

$$\mathbf{x} = \bar{\mathbf{x}}(\tau) + \mathbf{y} \quad (2.8)$$

Substituting (2.8) in (2.1) and expanding in Taylor series around  $\bar{\mathbf{x}}(\tau)$  we obtain

$$\mathbf{x}' = \bar{\mathbf{x}}'(\tau) + \mathbf{y}' = \mathbf{X}(\bar{\mathbf{x}}(\tau)) + \mathbf{DX}(\bar{\mathbf{x}}(\tau))\mathbf{y} + \mathcal{O}(\|\mathbf{y}\|^2) \quad (2.9)$$

where  $\mathbf{DX}$  is the derivative of  $\mathbf{X}$  represented in the canonical basis as the Jacobian matrix  $\mathbf{A}$  having components  $A_{i,j} = \frac{\partial X_i}{\partial x_j}$ ;  $\|\dots\|$  denotes a norm in  $\mathbb{R}^n$ . To derive (2.9) we have assumed that  $\mathbf{X}$  is at least of class  $C^2$ . Using the fact that  $\bar{\mathbf{x}}'(\tau) = \mathbf{X}(\bar{\mathbf{x}}(\tau))$  equation (2.9) is reduced to

$$\mathbf{y}' = \mathbf{DX}(\bar{\mathbf{x}}(\tau))\mathbf{y} + \mathcal{O}(\|\mathbf{y}\|^2) \quad (2.10)$$

Equation (2.10) describe the evolution of the orbits near to  $\bar{\mathbf{x}}(\tau)$ . For stability questions we will concentrate ourselves to the behavior of solutions arbitrarily close to  $\bar{\mathbf{x}}(\tau)$ , thus it seems reasonably to think that this question can be resolved studying the associated linear system

$$\mathbf{y}' = \mathbf{DX}(\bar{\mathbf{x}}(\tau))\mathbf{y}. \quad (2.11)$$

Hence, the question of the stability of  $\bar{\mathbf{x}}(\tau)$  involves the following two steps:

1. To determine if the trivial solution  $\mathbf{y} = \mathbf{0}$  of (2.11) is stable.
2. Prove that the stability (instability) of the solution  $\mathbf{y} = \mathbf{0}$  of (2.11) implies the stability (instability) of  $\bar{\mathbf{x}}(\tau)$ .

The first step is as difficult as to solve the original problem, since there not exist general analytical methods to find solutions of linear differential equations with variable

coefficients. However, there is an special type of solutions for which this problem can be easily resolved: equilibrium solutions.

Let us define an special type of solutions arising for autonomous vector fields (2.2).

**Definition 7** *A singular point of an autonomous vector field 2.2 is a point  $\bar{\mathbf{x}} \in \mathbb{R}^n$  such that  $\mathbf{X}(\bar{\mathbf{x}}) = \mathbf{0}$ .*

If  $\bar{\mathbf{x}}(\tau)$  is an equilibrium solution defined as  $\bar{\mathbf{x}}(\tau) = \bar{\mathbf{x}}$ , where  $\bar{\mathbf{x}}$  is a singular point of the vector field (2.2), then  $\mathbf{DX}(\bar{\mathbf{x}}(\tau)) = \mathbf{DX}(\bar{\mathbf{x}})$  is matrix with constant entries, and the solution  $\mathbf{y}$  of (2.11) passing through the point  $\mathbf{y}_0 \in \mathbb{R}^n$  at  $\tau = 0$  can be written immediately as

$$\mathbf{y}(\tau) = e^{\mathbf{DX}(\bar{\mathbf{x}})\tau} \mathbf{y}_0,$$

where for any  $\mathbf{A} \in M_n(\mathbb{R})$  (the vector space of real matrices of order  $n$ ), we have defined the “exponential” matrix

$$e^{\mathbf{A}\tau} = \mathbf{id} + \mathbf{A}\tau + \frac{1}{2!}\mathbf{A}^2\tau^2 + \cdots + \frac{1}{n!}\mathbf{A}^n\tau^n + \dots,$$

with  $\mathbf{id}$  denoting the identity matrix  $n \times n$ . Then, the solution  $\mathbf{y} = \mathbf{0}$  is asymptotically stable if all the eigenvalues of  $\mathbf{DX}(\bar{\mathbf{x}})$  have negative real parts. The answer to step 2 is given in the theorem 9 to be discussed next. Its proof require the use of the so-called Lyapunov functions. Thus, let us formulate the following stability criteria for autonomous systems due to Lyapunov.

**Theorem 6 (Theorem 2.0.1, [358])** *Consider the vector field (2.2). Let  $\bar{\mathbf{x}}$  be a singular point of (2.2) and let  $V : U \rightarrow \mathbb{R}$  a  $C^1$  function defined in a neighborhood  $U$  of  $\bar{\mathbf{x}}$  such that*

- i)  $V(\bar{\mathbf{x}}) = 0$  and  $V(\mathbf{x}) > 0$  if  $\mathbf{x} \neq \bar{\mathbf{x}}$
- ii)  $V'(\mathbf{x}) \equiv \nabla V \cdot \mathbf{x}' \leq 0$  in  $U \setminus \bar{\mathbf{x}}$  then  $\bar{\mathbf{x}}$  is stable.<sup>1</sup> Besides, if
- iii)  $V'(\mathbf{x}) < 0$  in  $U \setminus \bar{\mathbf{x}}$  then  $\bar{\mathbf{x}}$  is asymptotically stable.

The function  $V$  of the theorem 6 is referred as Lyapunov function. If in theorem 6 the condition iii) holds, then  $V$  is referred as strict Lyapunov function. If  $U = \mathbb{R}^n$ , then  $\bar{\mathbf{x}}$  is said to be globally asymptotically stable if i) and iii) in theorem 6 holds.

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<sup>1</sup>Here and through the text, the comma in  $V'(\mathbf{x})$  means total derivative with respect time. And it is referred as derivative through the flow-orbits, or Eulerian derivative.

### 2.2.3.1 Linear stability theorem

The Taylor expansion of  $\mathbf{X}(\mathbf{x}, \tau)$  in a neighborhood of  $\mathbf{x} = \mathbf{0}$  can be used to obtain a linear problem for small deviations of  $\mathbf{x} = \mathbf{0}$ .

Since  $\mathbf{X}(\mathbf{x}, \tau) = \mathbf{0}$ , when expanding in Taylor series in a neighborhood of the origin we obtain  $\mathbf{X}(\mathbf{x}, \tau) = \mathbf{A}(\tau)\mathbf{x} + \mathbf{G}(\mathbf{x}, \tau)$  where  $\mathbf{A}(\tau) = \mathbf{DX}(\mathbf{0}, \tau)$  is the jacobian matrix having components  $A_{i,j} = \frac{\partial X_i}{\partial x_j}(\mathbf{0}, \tau)$ ,  $i, j = 1 \dots n$ , and  $\mathbf{G}$  is the rest in the Taylor formula, such that for each  $\tau$  exists a constant  $K_1$  such that  $\|\mathbf{G}(\mathbf{x}, \tau)\| \leq K_1 \|\mathbf{x}\|^2$ . That is  $\mathbf{G}(\mathbf{x}, \tau) = \mathcal{O}(\|\mathbf{x}\|^2)$  as  $\mathbf{x} \rightarrow 0$  for each  $\tau$ . If the vector field  $\mathbf{X}$  is linear, then  $\mathbf{G} \equiv \mathbf{0}$ .

In the following we assume that the above estimate holds uniformly for  $\tau_0 \leq \tau < +\infty$  (i.e.,  $K_1$  does not depends on  $\tau$ ).

If the linear problem is exponentially asymptotically stable, then the solutions of the unperturbed system can be studied, in a neighborhood of  $\mathbf{x} = \mathbf{0}$ , using the linear problem and ignoring  $\mathbf{G}$ .

Let be the linear problem exponentially asymptotically stable and  $\Phi(\tau)$  denoting the fundamental solution of the linear problem

$$\Phi' = \mathbf{A}(\tau)\Phi, \Phi(\tau_0) = \mathbf{id}, \quad (2.12)$$

with  $\mathbf{id}$  denoting the identity matrix  $n \times n$ .

The connection between the linear problem and the nonlinear one can be established using the variation of constant formula and considering  $\mathbf{G}$  as a known matrix. Therefore, the initial value problem

$$\mathbf{x}' = \mathbf{A}(\tau)\mathbf{x} + \mathbf{G}(\mathbf{x}, \tau), \mathbf{x}(\tau_0) = \mathbf{x}_0, \quad (2.13)$$

is equivalent to the integral equation

$$\mathbf{x}(\tau) = \Phi(\tau)\mathbf{x}_0 + \int_{\tau_0}^{\tau} \Phi(\tau)\Phi(s)^{-1}\mathbf{G}(\mathbf{x}(s), s)ds. \quad (2.14)$$

The matrix  $\Phi$  encodes information about the behavior of the solutions of the linear problem and  $\mathbf{G}$  encodes information about the nonlinearity of the problem. As commented before the assumption of the exponential asymptotically stability of the linear problem will be crucial for the application of the linearization technique.

**Hypothesis H:** There exist positive constants  $K$  and  $\alpha$ , such that

$$\|\Phi(\tau)\Phi(s)^{-1}\| \leq Ke^{-\alpha(\tau-s)}, \forall \tau_0 \leq s \leq \tau < +\infty.$$

**Theorem 7 (Linear stability)** *If Hypothesis H holds, and  $\mathbf{G}(\mathbf{x}, \tau) = \mathcal{O}(\|\mathbf{x}\|^2)$  as  $\mathbf{x} \rightarrow 0$  uniformly for  $\tau_0 \leq \tau < +\infty$ . Then, there exists  $\delta_0 > 0$  such that, if  $\|\mathbf{x}(\tau_0)\| < \delta_0$ , then,*

exists a positive constant  $\alpha_0$  such that

$$\|\mathbf{x}(\tau)\| < K\|\mathbf{x}(\tau_0)\|e^{-\alpha_0(\tau-\tau_0)}$$

for all  $\tau \geq \tau_0$ . In this way the solution emerging from this initial state tends to  $\mathbf{x} = \mathbf{0}$  as  $\tau \rightarrow +\infty$ .

**Proof.** See [367], pp. 93-94.

Therefore, if the small perturbations problem is exponentially asymptotically stable, the nonlinear problem also is.

For autonomous systems the matrix  $\mathbf{A}$  have constant entries  $A_{i,j} = \frac{\partial X_i}{\partial x_j}(\mathbf{0})$ . In such case a sufficient condition for **Hypothesis H** (only valid for autonomous systems) is that

**Theorem 8 (Sufficient condition for Hypothesis H)** *Let  $\mathbf{A} \in M_n(\mathbb{R})$  (the vector space of real matrices of order  $n$ ) a constant matrix. If there exists a positive constant  $\alpha$ , such that the eigenvalues  $\lambda$  of  $\mathbf{A}$  satisfy  $\Re(\lambda) < -\alpha$ , then **Hypothesis H** holds.*

**Proof.** Let  $\mathbf{A} \in M_n(\mathbb{R})$  a constant matrix. Then each entry of the matrix  $e^{\mathbf{A}\tau}$  is a linear combination of the functions  $\tau^k e^{l\tau} \cos m\tau, \tau^k e^{l\tau} \sin m\tau$ , where  $l + mi$  denotes the eigenvalues of  $\mathbf{A}$  with  $m \geq 0$  ( $m = 0$  corresponds to real eigenvalues) and  $k$  is an integer taking values  $0, 1, 2, \dots, n-1$  less than the multiplicity of the corresponding eigenvalue ([348] p. 135). The fundamental solution of the linear part of the problem satisfies  $\Phi(\tau)\Phi(s)^{-1} = e^{\mathbf{A}(\tau-s)}$ , for all  $\tau_0 \leq s \leq \tau < +\infty$ . From the previous result and given the fact that for any  $\epsilon > 0$  and  $n > 0$ , there exists a constant  $C$  such that  $\tau^n < Ce^{\epsilon\tau}$ , for all  $\tau \geq \tau_0$  follows **Hypothesis H**.

Now let us formulate a version of the linear stability theorem 7 which is adequate to investigate the stability of singular points of autonomous systems.

**Theorem 9 (Theorem 1.2.5 [358])** *Suppose that the eigenvalues of  $\mathbf{DX}(\bar{\mathbf{x}})$  have negative real parts. Then, the equilibrium solution  $\mathbf{x} = \bar{\mathbf{x}}$  of the nonlinear vector field (2.2) is asymptotically stable.*

Sketch of the Proof. Express the the nonlinear vector field (2.2) in the form

$$\mathbf{y}' = \mathbf{DX}(\bar{\mathbf{x}})\mathbf{y} + \mathbf{R}(\mathbf{y}) \tag{2.15}$$

where  $\mathbf{y} = \mathbf{x} - \bar{\mathbf{x}}(\tau)$  and  $\mathbf{R}(\mathbf{y}) = \mathcal{O}(\|\mathbf{y}\|^2)$ .

Introduce the coordinate re-scaling

$$\mathbf{y} = \epsilon \mathbf{u}, \quad 0 < \epsilon < 1. \tag{2.16}$$

Then, taking an small  $\epsilon$  implies taking an small  $\mathbf{y}$ .

Under (2.16), the system (2.15) becomes

$$\mathbf{u}' = \mathbf{DX}(\bar{\mathbf{x}})\mathbf{u} + \bar{\mathbf{R}}(\mathbf{u}, \epsilon)$$

where  $\bar{\mathbf{R}}(\mathbf{u}, \epsilon) = \mathbf{R}(\epsilon\mathbf{u})/\epsilon$ . It is clear that  $\bar{\mathbf{R}}(\mathbf{u}, 0) = 0$  since  $\mathbf{R}(\mathbf{y}) = \mathcal{O}(\|\mathbf{y}\|^2)$ .

Choose as Lyapunov function the function

$$V(\mathbf{u}) = \frac{1}{2}\|\mathbf{u}\|^2.$$

Then

$$V'(\mathbf{u}) \equiv \nabla V(\mathbf{u})\mathbf{u}' = (\mathbf{u} \cdot \mathbf{DX}(\bar{\mathbf{x}})\mathbf{u}) + (\mathbf{u} \cdot \bar{\mathbf{R}}(\mathbf{u}, \epsilon)). \quad (2.17)$$

From linear algebra we have that if all the eigenvalues of  $\mathbf{DX}(\bar{\mathbf{x}})$  have negative real parts, then there exists a basis such that

$$(\mathbf{u} \cdot \mathbf{DX}(\bar{\mathbf{x}})\mathbf{u}) < K\|\mathbf{u}\|^2 \quad (2.18)$$

for some real number  $K$  and all  $\mathbf{u}$  (see [348]). Then, choosing  $\epsilon$  small enough, (2.17) will be strictly negative, which implies, using theorem 6, that the singular point  $\bar{\mathbf{x}}$  is asymptotically stable. To finish the proof one need to show that this result does not depends on the particular basis for which (2.18) holds.

## 2.2.4 Flow for autonomous vector fields

Qualitative analysis of a system begins with the location of singular points. Once the singular points of the vector field are obtained, it is of interest to consider the dynamics in a local neighborhood of each of the points. Assuming that the vector field  $\mathbf{X}(\mathbf{x})$  is of class  $C^1$  the process of determining the local behavior is based on the linear approximation of the vector field in the local neighborhood of the singular point  $\bar{\mathbf{x}}$ . In this neighborhood

$$\mathbf{X}(\mathbf{x}) \approx \mathbf{DX}(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}) \quad (2.19)$$

where  $\mathbf{DX}(\bar{\mathbf{x}})$  is the Jacobian of the vector field at the singular point  $\bar{\mathbf{x}}$ . The system (2.19) is referred to as the *linearization of the DE at the singular point*. Each of the singular points can then be classified according to the eigenvalues of the Jacobian of the linearized vector field at the point.

**Definition 8** *Let  $\bar{\mathbf{x}}$  be a singular point of the vector field 2.2. The point  $\bar{\mathbf{x}}$  is called a hyperbolic singular point if  $\Re(\lambda_i) \neq 0$  for all eigenvalues,  $\lambda_i$ , of the Jacobian of the vector field  $\mathbf{X}(\mathbf{x})$  evaluated at  $\bar{\mathbf{x}}$ . Otherwise the point is called non-hyperbolic.*

The notion of “hyperbolicity of a singular point” is defined in terms of the linearization around the singular point. This notion is extended to more general trajectories and to

invariant sets (manifolds). In all these cases the hyperbolicity is defined also in terms of the linearization around the trajectories or to the invariant sets (manifolds). Hyperbolicity persists under small perturbations of the vector field [358].

**Definition 9** *A set of non-isolated singular points is said to be normally hyperbolic if the only eigenvalues with zero real parts are those whose corresponding eigenvectors are tangent to the set.*

Since by definition any point on a set of non-isolated singular points will have at least one eigenvalue which is zero, all points in the set are *non-hyperbolic*. A set which is normally hyperbolic can, however, be completely classified as per it's stability by considering the signs of the eigenvalues in the remaining directions (i.e., for a curve, in the remaining  $n - 1$  directions) [368].

The classification then follows from the fact that if the singular point is hyperbolic in nature the flows of the non-linear system and it's linear approximation are *topologically equivalent* in a neighborhood of the singular point. This result is given in the form of the following theorem:

**Theorem 10 (Hartman-Grobman Theorem)** *Consider the vector field 2.2, where  $\mathbf{X}$  is of class  $C^1$ . If  $\bar{x}$  is a hyperbolic singular point of the 2.2 then there exists a neighborhood,  $\bar{U}$ , of  $\bar{x}$  on which the flow is topologically equivalent to the flow of the linearization of the DE at  $\bar{x}$ . That is, there exists an homeomorphism  $h : U \rightarrow \bar{U}$  defined in a neighborhood  $U$  of the origin such that  $h(e^{\mathbf{D}\mathbf{X}(\bar{x})\tau}y) = \mathbf{x}(\tau, \tau_0, h(y))$  for all  $y \in U, t \in \mathbb{R}$  (theorem 19.12.6 in [358] p. 350).*

**Proof.** See [349].

Given the autonomous vector field 2.2, without loss of generality we can assume that the solutions exists for all  $\tau \in \mathbb{R}$  (if not we can apply theorem 3 or 4 in order to do so). Thus we can define the concept of flow:

**Definition 10 (Flow for autonomous vector fields)** *Given the vector field 2.2, such that  $\mathbf{X}$  is of class  $C^1$ , and whose orbits are defined for all  $\tau \in \mathbb{R}$ . Let  $\mathbf{x}(\tau, \mathbf{x}_0)$  the unique maximal solution that satisfies  $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$ . The flow is defined as the one-parametric family of mappings  $\{\mathbf{g}^\tau\}_{\tau \in \mathbb{R}}$  such that  $\mathbf{g}^\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n, \mathbf{g}^\tau(\mathbf{y}) = \mathbf{x}(\tau, \mathbf{y})$  for all  $\mathbf{y} \in \mathbb{R}^n$  (definition 4.1, [1], p. 88).*

If the solutions of (2.2) are extensible as  $\tau \rightarrow +\infty$ , but not as  $\tau \rightarrow -\infty$ , we can define the positive semi-flow,  $\mathbf{g}^+_\tau$ , of the vector field replacing  $\tau \in \mathbb{R}$  by  $\tau \in \mathbb{R}^+$  in definition (10). Similarly, if the solutions of (2.2) are extensible as  $\tau \rightarrow -\infty$ , but not as  $\tau \rightarrow +\infty$ , we can define the negative semi-flow  $\mathbf{g}^-_\tau$ , of the vector field replacing  $\tau \in \mathbb{R}$  by  $\tau \in \mathbb{R}^-$  in definition (10).

The conceptual difference between a solution  $\mathbf{x}(\cdot, \mathbf{y})$  and a flow  $\mathbf{g}^\tau(\cdot)$  is in that

- For a fixed  $\mathbf{y} \in \mathbb{R}^n$ , the map  $\mathbf{x}(\cdot, \mathbf{y}) : \mathbb{R} \rightarrow \mathbb{R}^n$  represents the state,  $\mathbf{x}(\tau, \mathbf{y})$ , of the system for all  $\tau \in \mathbb{R}$ , such that  $\mathbf{x}(0, \mathbf{y}) = \mathbf{y}$  initially.
- For a fixed  $\tau \in \mathbb{R}$ , the map  $\mathbf{g}^\tau(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  represents the state,  $\mathbf{g}^\tau(\mathbf{y})$ , of the system at time  $\tau$  for all initial states  $\mathbf{y}$ .

**Theorem 11 (Smoothness of the flow)** *Given the vector field (2.2) such that  $\mathbf{X}$  is of class  $C^1$ . Then the flow of the vector field (2.2) consists of  $C^1$  mappings.*

From this result follows that the solutions of (2.2) are in a smooth dependence with respect to the initial conditions.

Some of the basic properties of a flow are the following

**Proposition 1 (Properties of the flow)** 1.  $\mathbf{g}^\tau(\mathbf{y})$  is  $C^r$  (see theorem 11).

2.  $\mathbf{g}^0(\mathbf{y}) = \mathbf{y}$ .

3.  $\mathbf{g}^{\tau+s}(\mathbf{y}) = \mathbf{g}^\tau(\mathbf{g}^s(\mathbf{y})) = \mathbf{g}^s(\mathbf{g}^\tau(\mathbf{y}))$ .

## 2.2.5 Invariant sets

Now let us introduce one of the fundamental concepts in the analysis of the properties of the flow associated to a vector field: the concept of invariant set.

**Definition 11 (Invariant set)** *Let  $S \subset \mathbb{R}^n$  be a set.  $S$  is called an invariant set under the vector field (2.2) if  $\mathbf{y} \in S$  implies  $\mathbf{x}(\tau, \mathbf{y}) \in S$  (where  $\mathbf{x}(0, \mathbf{y}) = \mathbf{y}$ ) for all  $\tau \in \mathbb{R}$ . If we consider the property valid for  $\tau \geq 0$  we say that  $S$  is positively invariant. On the other hand, if the property is valid for  $\tau \leq 0$  we say that  $S$  is negatively invariant.*

That is,  $S \subseteq \mathbb{R}^n$  is called an invariant set for the flow of the vector field (2.2) if for any point  $\mathbf{y} \in S$  the orbit through  $\mathbf{y}$  lies entirely in  $S$ , that is  $O(\mathbf{y}) \subseteq S$ . Succinctly, the invariant sets have the property that all trajectories initially at the invariant set, remain in the invariant set all past and future evolution. From the invariance of  $S$  under the flow of the vector field (2.2) follows that it acts as a dynamically independent object. Thus, when studying the dynamical properties of the flow of a vector field, we can investigate all possible invariant sets, and then, investigate the properties of the flow restricted to all of them.

We have a practical tool to determine some but not all the invariant sets of a vector field:

**Proposition 2 (Proposition 4.1, [1] p. 92)** *Let us consider the autonomous vector field (2.2) with flow  $\mathbf{g}^\tau$ . Let be defined a  $C^1$  function  $Z : \mathbb{R}^n \rightarrow \mathbb{R}$  which satisfies  $Z' \equiv \nabla Z \cdot \mathbf{X}(\mathbf{x}) = \alpha Z$  where  $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function. Then, the subsets of  $\mathbb{R}^n$  defined by  $\left\{ \mathbf{x} \in \mathbb{R}^n \mid Z(\mathbf{x}) \begin{smallmatrix} \leq \\ \geq \end{smallmatrix} 0 \right\}$  are invariant sets for the flow  $\mathbf{g}^\tau$ .*

**Definition 12 (Invariant Manifold)** *An invariant set  $S$  for the flow of the vector field (2.2) is called a  $C^r$  ( $r \geq 1$ ) invariant manifold if  $S$  has the structure of a differentiable  $C^r$  manifold. In that case, if the set  $S$  is positively (negatively) invariant, then it is called a positively (negatively) invariant manifold.*

The general concept of topological (resp., differentiable) manifold is given in definition 1 (resp., 6). Speaking without mathematical rigor, a manifold is a set that locally have the structure of an Euclidean space. In applications the manifolds are given more often as  $m$ -dimensional hypersurfaces immersed in  $\mathbb{R}^n$ . If the surface has not singular points, i.e., the derivative of the function representing the surface has maximal rank, then from the implicit function theorem follows that it can be represented locally as a graph. The surface is a manifold if the associated graph is  $C^r$ .

Another basic example of manifold is the following. Let  $\{s_1, \dots, s_n\}$  denoting the standard basis in  $\mathbb{R}^n$ . Let  $\{s_{i_1}, \dots, s_{i_j}\}$ ,  $j < n$  denoting any  $j$ -basis of vectors from this set; the set spanned by  $\{s_{i_1}, \dots, s_{i_j}\}$  forms a  $j$ -dimensional subspace of  $\mathbb{R}^n$  which is trivially a  $C^\infty$   $j$ -dimensional manifold. The main reason to choose these examples is in that, in the major part of our discussion whenever we use the term manifold, it will be sufficient to think in one of the following situations:

1. **Linear formulation:** a vector subspace of  $\mathbb{R}^n$
2. **Nonlinear formulation:** a surface immersed in  $\mathbb{R}^n$  which can be represented locally as a graph (this can be justified by means of the implicit function theorem).

### 2.2.5.1 Stable, Unstable and Center subspaces of singular points of linear autonomous vector fields

Now let us return to our study of the structure of the orbits near a singular point  $\mathbf{x} = \bar{\mathbf{x}}$  of (2.2) in order to describe some important invariant manifolds that arise in such investigation.

Let  $\bar{\mathbf{x}}$  a singular point of the vector field (2.2) defined in  $\mathbb{R}^n$ . Following the discussion in section 2.2.3, it is natural to investigate the associated linear system

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{2.20}$$

where  $\mathbf{A}$  is a matrix with constant coefficients  $\mathbf{D}\mathbf{X}(\bar{\mathbf{x}})$ .



It is a straightforward matter to show that if the eigenvalues of the matrix  $\mathbf{A}$  are all positive the solutions in the neighborhood of  $\bar{\mathbf{y}} = \mathbf{0}$  all diverge from that point. This point is then referred to as a source. Similarly, if the eigenvalues all have negative real parts all solutions converge to the singular point  $\bar{\mathbf{y}} = \mathbf{0}$ , and the point is referred to as a sink. Therefore, it follows from topological equivalence that if all eigenvalues of the Jacobian of the vector field for a non-linear system of ODEs have positive real parts the point is classified as a source (and all orbits diverge from the singular point), and if the eigenvalues all have negative real parts the point is classified as a sink.

In most cases the eigenvalues of the linearized system (2.19) will have eigenvalues with both positive, negative and/or zero real parts. In these cases it is important to identify which orbits are attracted to the singular point, and which are repelled away as the independent variable tends to infinity.

For a linear system of ODEs, (2.20), the phase space  $\mathbb{R}^n$  is spanned by the eigenvectors of  $\mathbf{A}$ . These eigenvectors divide the phase space into three distinct subspaces; namely:

$$\begin{aligned} \text{The stable subspace} \quad E^s &= \text{span}(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_s) \\ \text{The unstable subspace} \quad E^u &= \text{span}(\mathbf{e}_{s+1}, \mathbf{e}_{s+2}, \dots, \mathbf{e}_{s+u}) \\ \text{The center subspace} \quad E^c &= \text{span}(\mathbf{e}_{s+u+1}, \mathbf{e}_{s+u+2}, \dots, \mathbf{e}_{s+u+c}) \end{aligned}$$

where  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_s\}$  are the generalized eigenvectors of  $\mathbf{A}$  having associated eigenvalues with negative real part;  $\{\mathbf{e}_{s+1}, \mathbf{e}_{s+2}, \dots, \mathbf{e}_{s+u}\}$  are those whose eigenvalues have positive real part, and  $\{\mathbf{e}_{s+u+1}, \mathbf{e}_{s+u+2}, \dots, \mathbf{e}_{s+u+c}\}$  are those whose eigenvalues have zero eigenvalues. These ones are examples of invariant subspaces (manifolds), since the solutions of (2.20) with initial conditions entirely contained in  $E^s$ ,  $E^u$ , or  $E^c$  should be remain forever in this particular subspace. Besides, flows (or orbits) in the stable subspace asymptote in the future to the singular point, and those in the unstable subspace asymptote in the past to the singular point.

**Invariance of the stable, unstable and center manifold** Consider the matrix  $\mathbf{A}$  associated to the linear vector field (2.19) as a linear application from  $\mathbb{R}^n$  to itself. It is clear that  $E^s$ ,  $E^u$  and  $E^c$  are invariant subspaces under this linear application since each subspace is generated by a collection of generalized eigenvectors [358]. We want to prove that they are also invariant under the linear application  $e^{\mathbf{A}\tau}$ .

Suppose that  $V \subset \mathbb{R}^n$  is an invariant set under the linear map  $\mathbf{A}$ . Then,

- For each  $c \in \mathbb{R}$ ,  $V$  is invariant with respect to  $c\mathbf{A}$ .
- For each integer  $n > 1$ ,  $V$  is invariant with respect to  $\mathbf{A}^n$ .
- Suppose that  $\mathbf{A}_1$  and  $\mathbf{A}_2$  are linear maps that leave  $V$  invariant, then  $V$  is invariant with respect to  $\mathbf{A}_1 + \mathbf{A}_2$ . This results follows for a finite number of linear applications  $\mathbf{A}_i$  that leaves  $V$  invariant.

Using all of these points follows that  $V$  is invariant under the linear application

$$L_n(\tau) \equiv \mathbf{id} + \mathbf{A}\tau + \frac{1}{2!}\mathbf{A}^2\tau^2 + \cdots + \frac{1}{n!}\mathbf{A}^n\tau^n = \sum_{i=0}^n \frac{1}{i!}\mathbf{A}^i\tau^i,$$

for each  $n$ , where  $\mathbf{id}$  is the  $n \times n$  identity matrix.

Using the fact that  $V$  is closed and that  $L_n(\tau)$  converges uniformly to  $e^{\mathbf{A}\tau}$ , we conclude that  $V$  is invariant under  $e^{\mathbf{A}\tau}$ .

Now let us discuss several examples.

In the example example 3.1.1 in [358], the eigenvalues of  $\mathbf{A}$  are reals and different, denoted by  $\lambda_1, \lambda_2 < 0, \lambda_3 > 0$ . Then,  $\mathbf{A}$  have three linearly independent (l.i) eigenvectors,  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  corresponding respectively to  $\lambda_1, \lambda_2, \lambda_3$ . Let us define the  $3 \times 3$  matrix  $\mathbf{T}$  taking as columns the eigenvectors

$$\mathbf{T} \equiv \begin{pmatrix} \vdots & \vdots & \vdots \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \vdots & \vdots & \vdots \end{pmatrix}, \quad (2.21)$$

Then, we have  $\mathbf{A} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}$  where

$$\mathbf{\Lambda} \equiv \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad (2.22)$$

Recall that the solution of (2.20) passing through  $\mathbf{y}_0 \in \mathbb{R}^n$  at  $\tau = 0$  is given by  $\mathbf{y}(\tau) = e^{\mathbf{A}\tau}\mathbf{y}_0 = e^{\mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}\tau}\mathbf{y}_0$ .

It is easy to see that

$$\mathbf{y}(\tau) = \begin{pmatrix} \vdots & \vdots & \vdots \\ \mathbf{e}_1 e^{\lambda_1 \tau} & \mathbf{e}_2 e^{\lambda_2 \tau} & \mathbf{e}_3 e^{\lambda_3 \tau} \\ \vdots & \vdots & \vdots \end{pmatrix} \mathbf{T}^{-1}\mathbf{y}_0. \quad (2.23)$$

According to the previous results we have that  $E^s = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$  and  $E^u = \text{span}(\mathbf{e}_3)$ . To see illustrate the invariance we choose a point  $\mathbf{y}_0 \in \mathbb{R}^3$ . Then,  $\mathbf{T}^{-1}$  induce a coordinate transformation that change the coordinates of  $\mathbf{y}_0$  with respect to the standard basis in  $\mathbb{R}^3$  (i.e.,  $(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T$ ) in coordinates with respect to the eigenbasis  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ .

Then, for  $\mathbf{y}_0 \in E^s$ ,

$$\mathbf{T}^{-1}\mathbf{y}_0 = \begin{pmatrix} z_{01} \\ z_{02} \\ 0 \end{pmatrix} \quad (2.24)$$

and for  $\mathbf{y}_0 \in E^u$ ,

$$\mathbf{T}^{-1}\mathbf{y}_0 = \begin{pmatrix} 0 \\ 0 \\ z_{03} \end{pmatrix} \quad (2.25)$$

Hence, by substituting (2.24) (resp., (2.25)) in (2.23), it is easy to see that  $\mathbf{y}_0 \in E^s$  (resp.,  $\mathbf{y}_0 \in E^u$ ) implies  $e^{\mathbf{A}\tau}\mathbf{y}_0 \in E^s$  (resp.,  $e^{\mathbf{A}\tau}\mathbf{y}_0 \in E^u$ ). Then,  $E^s$  and  $E^u$  are invariant manifolds. Also, for each  $\mathbf{y}_0 \in E^s$  (resp.,  $\mathbf{y}_0 \in E^u$ ) we have  $e^{\mathbf{A}\tau}\mathbf{y}_0 \rightarrow \mathbf{0}$  as  $\tau \rightarrow +\infty$  (resp.,  $\tau \rightarrow -\infty$ ).

In the example 3.1.2 in [358] the matrix  $\mathbf{A}$  have two complex conjugated eigenvalues  $\varrho \pm i\varpi$ ,  $\varrho < 0$ ,  $\varphi \neq 0$  and a real value  $\lambda > 0$ . Then,  $\mathbf{A}$  have three real generalized eigenvectors,  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  which can be used as columns of the transformation matrix  $\mathbf{T}$  such that

$$\Lambda \equiv \begin{pmatrix} \varrho & \varpi & 0 \\ -\varpi & \varrho & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}. \quad (2.26)$$

In this example

$$\mathbf{y}(\tau) = \mathbf{T}e^{\mathbf{A}\tau}\mathbf{T}^{-1}\mathbf{y}_0 = \mathbf{T} \begin{pmatrix} e^{\varrho\tau} \cos \varpi\tau & e^{\varrho\tau} \sin \varpi\tau & 0 \\ -e^{\varrho\tau} \sin \varpi\tau & e^{\varrho\tau} \cos \varpi\tau & 0 \\ 0 & 0 & e^{\lambda\tau} \end{pmatrix} \mathbf{T}^{-1}\mathbf{y}_0 \quad (2.27)$$

Using the arguments given in the example 3.1.2, it is clear that  $E^s = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$  is an invariant manifold of solutions exponentially decaying to zero as  $\tau \rightarrow +\infty$  and  $E^u = \text{span}(\mathbf{e}_3)$  is an invariant manifold of solutions exponentially decaying to zero as  $\tau \rightarrow -\infty$ .

In the example 3.1.3 in [358] the matrix  $\mathbf{A}$  have two real repeated eigenvalues,  $\lambda < 0$ , and a third distinct eigenvalue  $\gamma > 0$  such that there exist three real generalized eigenvectors,  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  which can be used as columns of the transformation matrix  $\mathbf{T}$  such that  $\mathbf{A}$  is transformed according to

$$\Lambda \equiv \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \gamma \end{pmatrix} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}. \quad (2.28)$$

Following the ideas of the previous examples, in this case, the solution passing through the point  $\mathbf{y}_0$  at time  $\tau = 0$  is given by

$$\mathbf{y}(\tau) = \mathbf{T}e^{\mathbf{A}\tau}\mathbf{T}^{-1}\mathbf{y}_0 = \mathbf{T} \begin{pmatrix} e^{\lambda\tau} & \tau e^{\lambda\tau} \sin \varpi\tau & 0 \\ 0 & e^{\lambda\tau} & 0 \\ 0 & 0 & e^{\gamma\tau} \end{pmatrix} \mathbf{T}^{-1}\mathbf{y}_0 \quad (2.29)$$

Using the same arguments as in example 3.1.1, it is clear that  $E^s = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$  is an invariant manifold of solutions decaying to zero as  $\tau \rightarrow +\infty$  and  $E^u = \text{span}(\mathbf{e}_3)$  is an invariant manifold of solutions decaying to zero as  $\tau \rightarrow -\infty$ .

In the non-linear case, the topological equivalence of flows allows for a similar classification of the singular points. The equivalence only applies in directions where the eigenvalue has non-zero real parts. In these directions, since the flows are topologically equivalent, there is a flow *tangent* to the eigenvectors.

Unlike a linear system of ODEs, a non-linear system allows for singular structures which are more complicated than that of the singular points, fixed lines or periodic orbits. These structures include, though are not limited to, such things as heteroclinic and/or homoclinic orbits and non-linear invariant sub-manifolds (for definitions see [358]).

### 2.2.5.2 Stable, Unstable and Center manifolds of singular points of nonlinear autonomous vector fields

It is well-known that a nonlinear autonomous vector field can be expressed locally in a neighborhood of a singular point,  $\bar{\mathbf{x}}$ , as

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{R}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^n, \quad (2.30)$$

where  $\mathbf{A} = \mathbf{D}\mathbf{X}(\bar{\mathbf{x}})$ , and  $\mathbf{R}(\mathbf{y}) = \mathcal{O}(\|\mathbf{y}\|^2)$ .

Using elementary algebra [348] follows that there exists a lineal transformation,  $\mathbf{T}$ , such that the linear part in (2.30),  $\mathbf{y}' = \mathbf{A}\mathbf{y}$ , can be expressed in the real Jordan form

$$\begin{aligned} \mathbf{u}' &= \mathbf{A}_s \mathbf{u}, \\ \mathbf{v}' &= \mathbf{A}_u \mathbf{v}, \\ \mathbf{w}' &= \mathbf{A}_c \mathbf{w}, \end{aligned} \quad (2.31)$$

where

$$\mathbf{T}^{-1}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) \equiv (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c, \quad s + u + c = n;$$

$\mathbf{A}_s$  is the  $s \times s$  matrix having eigenvalues with negative real parts;  $\mathbf{A}_u$  is the  $u \times u$  matrix having eigenvalues with positive real parts; and  $\mathbf{A}_c$  is the  $c \times c$  matrix having eigenvalues with zero real parts. By the change of coordinates induced by  $\mathbf{T}$  the nonlinear vector field (2.2) can be expressed as

$$\begin{aligned} \mathbf{u}' &= \mathbf{A}_s \mathbf{u} + \mathbf{R}_s(\mathbf{u}, \mathbf{v}, \mathbf{w}), \\ \mathbf{v}' &= \mathbf{A}_u \mathbf{v} + \mathbf{R}_u(\mathbf{u}, \mathbf{v}, \mathbf{w}), \\ \mathbf{w}' &= \mathbf{A}_c \mathbf{w} + \mathbf{R}_c(\mathbf{u}, \mathbf{v}, \mathbf{w}), \end{aligned} \quad (2.32)$$

where  $\mathbf{R}_s(\mathbf{u}, \mathbf{v}, \mathbf{w})$ ,  $\mathbf{R}_u(\mathbf{u}, \mathbf{v}, \mathbf{w})$ ,  $\mathbf{R}_c(\mathbf{u}, \mathbf{v}, \mathbf{w})$ , are, respectively the first  $s$ ,  $u$  and  $c$  com-

ponents of the vector field  $\mathbf{T}^{-1}\mathbf{R}(\mathbf{T}\mathbf{y})$ .

Let us consider the linear vector field (2.31). Following the previous discussion, the origin of (2.31) have a  $s$ -dimensional stable invariant manifold; a  $u$ -dimensional unstable invariant manifold; and a  $c$ -dimensional center invariant manifold, all of them intersecting the origin. The following theorem shows how the structure of the invariant subspaces of the origin change when passing from the study of the linear system (2.31) to nonlinear one (2.32).

**Theorem 12 (Local stable, unstable, and center manifolds at the origin)** *If (2.32) is of class  $C^r$ ,  $r \geq 2$ , then the singular point  $(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbf{0}$  of (2.32) have a local invariant stable manifold of dimension  $s$ ,  $W_{loc}^s(\mathbf{0})$ ; a local invariant unstable manifold of dimension  $u$ ,  $W_{loc}^u(\mathbf{0})$ ; and a local invariant center manifold of dimension  $c$ ,  $W_{loc}^c(\mathbf{0})$ , all of them intersecting at the origin. These manifolds are tangent at the origin to the respective invariant subspaces of the linear vector field (2.31). Then they can expressed locally as the graphs*

$$\begin{aligned} W_{loc}^s(\mathbf{0}) &= \{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c \mid \mathbf{v} = \mathbf{h}_{\mathbf{v}}^s(\mathbf{u}), \mathbf{w} = \mathbf{h}_{\mathbf{w}}^s(\mathbf{u}), \|\mathbf{u}\| < \delta, \\ &\quad \mathbf{h}_{\mathbf{v}}^s(\mathbf{0}) = \mathbf{0}, \mathbf{h}_{\mathbf{w}}^s(\mathbf{0}) = \mathbf{0}, \mathbf{Dh}_{\mathbf{v}}^s(\mathbf{0}) = \mathbf{0}, \mathbf{Dh}_{\mathbf{w}}^s(\mathbf{0}) = \mathbf{0}\}; \\ W_{loc}^u(\mathbf{0}) &= \{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c \mid \mathbf{u} = \mathbf{h}_{\mathbf{u}}^u(\mathbf{v}), \mathbf{w} = \mathbf{h}_{\mathbf{w}}^u(\mathbf{v}), \|\mathbf{v}\| < \delta, \\ &\quad \mathbf{h}_{\mathbf{u}}^u(\mathbf{0}) = \mathbf{0}, \mathbf{h}_{\mathbf{w}}^u(\mathbf{0}) = \mathbf{0}, \mathbf{Dh}_{\mathbf{u}}^u(\mathbf{0}) = \mathbf{0}, \mathbf{Dh}_{\mathbf{w}}^u(\mathbf{0}) = \mathbf{0}\}; \\ W_{loc}^c(\mathbf{0}) &= \{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c \mid \mathbf{u} = \mathbf{h}_{\mathbf{u}}^c(\mathbf{w}), \mathbf{v} = \mathbf{h}_{\mathbf{v}}^c(\mathbf{w}), \|\mathbf{w}\| < \delta, \\ &\quad \mathbf{h}_{\mathbf{u}}^c(\mathbf{0}) = \mathbf{0}, \mathbf{h}_{\mathbf{v}}^c(\mathbf{0}) = \mathbf{0}, \mathbf{Dh}_{\mathbf{u}}^c(\mathbf{0}) = \mathbf{0}, \mathbf{Dh}_{\mathbf{v}}^c(\mathbf{0}) = \mathbf{0}\}, \end{aligned} \quad (2.33)$$

where the functions  $\mathbf{h}_{\mathbf{v}}^s$ ,  $\mathbf{h}_{\mathbf{w}}^s$ ,  $\mathbf{h}_{\mathbf{u}}^u$ ,  $\mathbf{h}_{\mathbf{w}}^u$ ,  $\mathbf{h}_{\mathbf{u}}^c$ , and  $\mathbf{h}_{\mathbf{v}}^c$  are  $C^r$ -functions and  $\delta$  a positive small enough number. The orbits at  $W_{loc}^s(\mathbf{0})$  and at  $W_{loc}^u(\mathbf{0})$  have the same asymptotic properties as the orbits in the invariant subsets  $E^s$  and  $E^u$  respectively. That is, the orbits of (2.32) with initial conditions at  $W_{loc}^s(\mathbf{0})$  (resp.,  $W_{loc}^u(\mathbf{0})$ ) tends asymptotically to the origin at an exponential rate as  $\tau \rightarrow +\infty$  (resp.,  $\tau \rightarrow -\infty$ ) [theorem 3.2.1 in [358]].

**Proof.** See [369, 370, 371]

Stable, unstable or center manifold should be referred to something (e.g., singular point, set, etc.) in order to be meaningful.

The conditions  $\mathbf{Dh}_{\mathbf{v}}^s(\mathbf{0}) = \mathbf{0}$ ,  $\mathbf{Dh}_{\mathbf{w}}^s(\mathbf{0}) = \mathbf{0}$ , ... reflect the fact that the nonlinear manifolds are tangent to the associated invariant linear subspaces at the origin.

In the formulation of theorem 12, in expressions like “local invariant stable manifold ...”, the term “local” is referred to the fact that the manifolds are defined as a graph only in an small neighborhood of the singular point. Consequently, all these invariant manifolds have a boundary. Hence, they are only locally invariant in the sense that the orbits initially on them can abandon the local manifold, but only crossing the boundary. The invariance maintains because the vector field is tangent to the manifolds.

In case that the singular point is hyperbolic (i.e.,  $E^c = \emptyset$ ), the interpretation of theorem 12 is that the trajectories of the nonlinear vector field have qualitatively the same behavior of the orbits of the linear associated problem in a neighborhood of the singular point. This fact is explicitly expressed in theorem 10.

The stable and unstable manifolds are unique. This can be proved using a contraction mapping argument. For the center manifold, due to the non-hyperbolicity, the analysis is more difficult, and in general the center manifold is not unique. However, the center manifold is unique in all the orders in its Taylor expansion. That is, the all possible invariant manifolds differ only on small exponential perturbations depending on the distance from the origin to the singular point (see [358]).

It is important to note, however, that unlike the case of a linear system, the center manifold,  $W_{\text{loc}}^c(\mathbf{0})$  will contain all those dynamics not classified by linearization (i.e., the non-hyperbolic directions). In particular, this manifold may contain regions which are stable, unstable or neutral. The classification of the dynamics in this manifold can only be determined by utilizing more sophisticated methods, such as center manifold theorems or the theory of normal forms (see [358]).

### 2.2.5.3 Center Manifold Theory

In this section we offer the main techniques for the construction of center manifolds for vector fields in  $\mathbb{R}^n$ . We follow the approach in [358] chapter 18.

The setup is as follows. We consider vector fields in the form

$$\begin{aligned} \mathbf{x}' &= \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}), \\ \mathbf{y}' &= \mathbf{B}\mathbf{x} + \mathbf{g}(\mathbf{x}, \mathbf{y}), \quad (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^c \times \mathbb{R}^s, \end{aligned} \quad (2.34)$$

where

$$\begin{aligned} \mathbf{f}(\mathbf{0}, \mathbf{0}) &= \mathbf{0}, \mathbf{D}\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}, \\ \mathbf{g}(\mathbf{0}, \mathbf{0}) &= \mathbf{0}, \mathbf{D}\mathbf{g}(\mathbf{0}, \mathbf{0}) = \mathbf{0}. \end{aligned} \quad (2.35)$$

In the above,  $\mathbf{A}$  is a  $c \times c$  matrix having eigenvalues with zero real parts,  $\mathbf{B}$  is an  $s \times s$  matrix having eigenvalues with negative real parts, and  $\mathbf{f}$  and  $\mathbf{g}$  are  $C^r$  functions ( $r \geq 2$ ).

**Definition 13 (Center Manifold)** *An invariant manifold will be called a center manifold for (2.34) if it can locally be represented as follows*

$$W^c(\mathbf{0}) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^c \times \mathbb{R}^s : \mathbf{y} = \mathbf{h}(\mathbf{x}), |\mathbf{x}| < \delta\}; \quad \mathbf{h}(\mathbf{0}) = \mathbf{0}, D\mathbf{h}(\mathbf{0}) = \mathbf{0},$$

for  $\delta$  sufficiently small (cf. [358] p. 246, [372], p. 155).

The conditions  $\mathbf{h}(\mathbf{0}) = \mathbf{0}$ ,  $D\mathbf{h}(\mathbf{0}) = \mathbf{0}$  imply that  $W^c(\mathbf{0})$  is tangent to  $E^c$  at  $(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{0})$ , where  $E^c$  is the generalized eigenspace whose corresponding eigenvalues have zero real parts. The following three theorems (see theorems 18.1.2, 18.1.3 and 18.1.4 in [358] p. 245-248) are the main results to the treatment of center manifolds. The first two are existence and stability theorems of the center manifold for (2.34) at the origin. The third theorem allows to compute the center manifold to any desired degree accuracy by using Taylor series to solve a quasilinear partial differential equation that  $\mathbf{h}(\mathbf{x})$  must satisfy. The proof of those results is given in [373].

**Theorem 13 (Existence)** *There exists a  $C^r$  center manifold for (2.34). The dynamics of (2.34) restricted to the center manifold is, for  $\mathbf{u}$  sufficiently small, given by the following  $c$ -dimensional vector field*

$$\mathbf{u}' = \mathbf{A}\mathbf{u} + \mathbf{f}(\mathbf{u}, \mathbf{h}(\mathbf{u})), \quad \mathbf{u} \in \mathbb{R}^c. \quad (2.36)$$

The next results implies that the dynamics of (2.36) near  $\mathbf{u} = \mathbf{0}$  determine the dynamics of (2.34) near  $(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{0})$  (see also Theorem 3.2.2 in [374]).

**Theorem 14 (Stability)** *i) Suppose the zero solution of (2.36) is stable (asymptotically stable) (unstable); then the zero solution of (2.34) is also stable (asymptotically stable) (unstable). Then if  $(\mathbf{x}(\tau), \mathbf{y}(\tau))$  is a solution of (2.34) with  $(\mathbf{x}(0), \mathbf{y}(0))$  sufficiently small, then there is a solution  $\mathbf{u}(\tau)$  of (2.36) such that, as  $\tau \rightarrow \infty$*

$$\begin{aligned} \mathbf{x}(\tau) &= \mathbf{u}(\tau) + \mathcal{O}(e^{-r\tau}), \\ \mathbf{y}(\tau) &= \mathbf{h}(\mathbf{u}(\tau)) + \mathcal{O}(e^{-r\tau}), \end{aligned}$$

where  $r > 0$  is a constant.

**Dynamics Captured by the center manifold** Stated in words, this theorem says that for initial conditions of the *full system* sufficiently close to the origin, trajectories through them asymptotically approach a trajectory on the center manifold. In particular, singular points sufficiently close to the origin, sufficiently small amplitude periodic orbits, as well as small homoclinic and heteroclinic orbits are contained in the center manifold.

The obvious question now is how to compute the center manifold so that we can use the result of theorem 14? To answer this question we will derive an equation that  $\mathbf{h}(\mathbf{x})$  must satisfy in order to its graph to be a center manifold for (2.34).

Suppose we have a center manifold

$$W^c(\mathbf{0}) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^c \times \mathbb{R}^s : \mathbf{y} = \mathbf{h}(\mathbf{x}), |\mathbf{x}| < \delta\}; \quad \mathbf{h}(\mathbf{0}) = \mathbf{0}, D\mathbf{h}(\mathbf{0}) = \mathbf{0},$$

with  $\delta$  sufficiently small. Using the invariance of  $W^c(\mathbf{0})$  under the dynamics of (2.34), we derive a quasilinear partial differential equation that  $\mathbf{h}(\mathbf{x})$  must satisfy. This is done as follows:

1. The  $(\mathbf{x}, \mathbf{y})$  coordinates of any point on  $W^c(\mathbf{0})$  must satisfy

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) \quad (2.37)$$

2. Differentiating (2.37) with respect to time implies that the  $(\mathbf{x}', \mathbf{y}')$  coordinates of any point on  $W^c(\mathbf{0})$  must satisfy

$$\mathbf{y}' = D\mathbf{h}(\mathbf{x}) \mathbf{x}' \quad (2.38)$$

3. Any point in  $W^c(\mathbf{0})$  obey the dynamics generated by (2.34). Therefore substituting

$$\begin{aligned} \mathbf{x}' &= \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x})), \\ \mathbf{y}' &= \mathbf{B}\mathbf{h}(\mathbf{x}) + \mathbf{g}(\mathbf{x}, \mathbf{h}(\mathbf{x})) \end{aligned}$$

into (2.38) gives

$$\mathcal{N}(\mathbf{h}(\mathbf{x})) \equiv D\mathbf{h}(\mathbf{x}) [\mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x}))] - \mathbf{B}\mathbf{h}(\mathbf{x}) - \mathbf{g}(\mathbf{x}, \mathbf{h}(\mathbf{x})) = 0. \quad (2.39)$$

Equation (2.39) is a quasilinear partial differential that  $\mathbf{h}(\mathbf{x})$  must satisfy in order for its graph to be an invariant center manifold. To find the center manifold, all we need to do is solve (2.39).

Unfortunately, it is probably more difficult to solve (2.39) than our original problem; however the following theorem give us a method for computing an approximated solution of (2.39) to any desired degree of accuracy.

**Theorem 15 (Approximation)** *Let  $\Phi : \mathbb{R}^c \rightarrow \mathbb{R}^s$  be a  $C^1$  mapping with  $\Phi(\mathbf{0}) = \mathbf{0}$  and  $D\Phi(\mathbf{0}) = \mathbf{0}$  such that  $\mathcal{N}(\Phi(\mathbf{x})) = \mathcal{O}(\|\mathbf{x}\|^q)$  as  $\mathbf{x} \rightarrow \mathbf{0}$  for some  $q > 1$ . Then,  $|\mathbf{h}(\mathbf{x}) - \Phi(\mathbf{x})| = \mathcal{O}(\|\mathbf{x}\|^q)$  as  $\mathbf{x} \rightarrow \mathbf{0}$*

This theorem allows us to compute the center manifold to any desired degree of accuracy by solving (2.39) to the same degree of accuracy. For this task power series expansions will work nicely. Let us consider a concrete example further in section 3.6.3.1.

#### 2.2.5.4 Normal Forms

In this section we offer the main techniques for the construction of normal forms for vector fields in  $\mathbb{R}^n$ . We follow the approach in [359].



Let  $\mathbf{X} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a smooth vector field satisfying  $\mathbf{X}(\mathbf{0}) = \mathbf{0}$ . We can formally construct the Taylor expansion of  $\mathbf{x}$  around  $\mathbf{0}$ , namely,  $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \dots + \mathbf{X}_k + O(\|\mathbf{x}\|^{k+1})$ , where  $\mathbf{X}_r \in H^r$ , the real vector space of vector fields whose components are homogeneous polynomials of degree  $r$ . For  $r = 1$  to  $k$  we write

$$\begin{aligned} \mathbf{X}_r(\mathbf{x}) &= \sum_{m_1=1}^r \dots \sum_{m_n=1}^r \sum_{j=1}^n \mathbf{X}_{\mathbf{m},j} \mathbf{x}^{\mathbf{m}} \mathbf{e}_j, \\ \sum_i m_i &= r, \end{aligned} \quad (2.40)$$

Observe that  $\mathbf{X}_1 = \mathbf{DX}(\mathbf{0})\mathbf{x} \equiv \mathbf{Ax}$ , i.e., the Jacobian matrix.

The aim of the normal form calculation is to construct a sequence of transformations which successively remove the non-linear term  $\mathbf{X}_r$ , starting from  $r = 2$ .

The transformation themselves are of the form

$$\mathbf{x} = \mathbf{y} + \mathbf{h}_r(\mathbf{y}), \quad (2.41)$$

where  $\mathbf{h}_r \in H^r$ ,  $r \geq 2$ .

The effect of (2.41) in  $\mathbf{X}_1$  is as follows [359]: Observe that  $\mathbf{x} = O(\|\mathbf{y}\|)$ . Then, the inverse of (2.41) takes the form

$$\mathbf{y} = \mathbf{x} - \mathbf{h}_r(\mathbf{x}) + O(\|\mathbf{x}\|^{r+1}). \quad (2.42)$$

By applying total derivatives in both sides, and assuming  $\mathbf{x}' = \mathbf{Ax} + \mathbf{X}_r(\mathbf{x})$ , we find

$$\mathbf{y}' = \mathbf{Ay} - \mathbf{L}_\mathbf{A} \mathbf{h}_r(\mathbf{y}) + \mathbf{X}_r(\mathbf{y}) + O(\|\mathbf{y}\|^{r+1}) \quad (2.43)$$

where  $\mathbf{L}_\mathbf{A}$  is the linear operator that assigns to  $\mathbf{h}(\mathbf{y}) \in H^r$  the Lie bracket of the vector fields  $\mathbf{Ay}$  and  $\mathbf{h}(\mathbf{y})$ :

$$\begin{aligned} \mathbf{L}_\mathbf{A} : H^r &\rightarrow H^r \\ \mathbf{h} &\rightarrow \mathbf{L}_\mathbf{A} \mathbf{h}(\mathbf{y}) = \mathbf{Dh}(\mathbf{y})\mathbf{Ay} - \mathbf{Ah}(\mathbf{y}). \end{aligned} \quad (2.44)$$

Both  $\mathbf{L}_\mathbf{A}$  and  $\mathbf{X}_r \in H^r$ , so that the deviation of the right-hand side of (2.43) from  $\mathbf{Ay}$  has no terms of order less than  $r$  in  $\|\mathbf{y}\|$ . This means that if  $\mathbf{X}$  is such that  $\mathbf{X}_2 = \dots \mathbf{X}_{r-1} = 0$ , they will remain zero under the transformation (2.41). This makes clear how we may be able to remove  $\mathbf{X}_r$  from a suitable choice of  $\mathbf{h}_r$ .

The proposition 2.3.2 in [359] states that if the inverse of  $\mathbf{L}_\mathbf{A}$  exists, the differential equation

$$\mathbf{x}' = \mathbf{Ax} + \mathbf{X}_r(\mathbf{x}) + O(\|\mathbf{x}\|^{r+1}) \quad (2.45)$$

with  $\mathbf{X}_r \in H^r$ , can be transformed to

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + O(\|\mathbf{y}\|^{r+1}) \quad (2.46)$$

by the transformation (2.41) where

$$\mathbf{h}_r(\mathbf{y}) = \mathbf{L}_\mathbf{A}^{-1} \mathbf{X}_r(\mathbf{y}) \quad (2.47)$$

The equation

$$\mathbf{L}_\mathbf{A} \mathbf{h}_r(\mathbf{y}) = \mathbf{X}_r(\mathbf{y}) \quad (2.48)$$

is named the homological equation.

If  $\mathbf{A}$  has distinct eigenvalues  $\lambda_i$ ,  $i = 1, 2, 3$ , its eigenvectors form a basis of  $\mathbb{R}^n$ . Relative to this eigenbasis,  $\mathbf{A}$  is diagonal. It can be proved (see proof in [359]) that  $\mathbf{L}_\mathbf{A}$  has eigenvalues  $\Lambda_{\mathbf{m},i} = \mathbf{m} \cdot \lambda - \lambda_i = \sum_j m_j \lambda_j - \lambda_i$  with associated eigenvectors  $\mathbf{x}^{\mathbf{m}} \mathbf{e}_i$ . The operator,  $\mathbf{L}_\mathbf{A}^{-1}$ , exists if and only if the  $\Lambda_{\mathbf{m},i} \neq 0$ , for every allowed  $\mathbf{m}$  and  $i = 1 \dots n$ .

If we were able to remove all the nonlinear terms in this way, then the vector field can be reduced to its linear part

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}) \rightarrow \mathbf{y}' = \mathbf{A}\mathbf{y}.$$

Unfortunately, not all the higher order terms vanishes by applying these transformations. It is the case if resonance occurs.

The  $n$ -tuple of eigenvalues  $\lambda = (\lambda_1, \dots, \lambda_n)^T$  is resonant of order  $r$  (see definition 2.3.1 in [359]) if there exist some  $\mathbf{m} = (m_1, m_2, \dots, m_n)^T$  (a  $n$ -tuple of non-negative integers) with  $m_1 + m_2 + \dots + m_n = r$  and some  $i = 1 \dots n$  such that  $\lambda_i = \mathbf{m} \cdot \lambda$ , i.e., if  $\Lambda_{\mathbf{m},i} = 0$  for some  $\mathbf{m}$  and some  $i$ .

If there is no resonant eigenvalues, and provided they are different, we can use the eigenvectors of  $\mathbf{A}$  as a basis for  $H^r$ . Then, we can write  $\mathbf{h}_r$  as

$$\mathbf{h}_r(\mathbf{x}) = \sum_{\mathbf{m}, i, \sum m_j = r} h_{\mathbf{m},i} \mathbf{x}^{\mathbf{m}} \mathbf{e}_i$$

and any vector field  $\mathbf{X} \in H^r$  as

$$\mathbf{X}(\mathbf{x}) = \sum_{\mathbf{m}, i, \sum m_j = r} \mathbf{X}_{\mathbf{m},i} \mathbf{x}^{\mathbf{m}} \mathbf{e}_i$$

where  $\mathbf{m} = (m_1, m_2, \dots, m_n)^T$ ,  $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$  and  $\mathbf{e}_i$ ,  $i = 1, \dots, n$  stands for the canonical basis in  $\mathbb{R}^n$ . If the eigenvalues of  $\mathbf{A}$  are not resonant of order  $r$ , then

$$h_{\mathbf{m},i} = \mathbf{X}_{\mathbf{m},i} / \Lambda_{\mathbf{m},i}.$$

This gives  $\mathbf{h}_r$  explicitly in terms of  $\mathbf{X}_r$ .

In case that resonance occurs, we proceed as follows. If  $\mathbf{A}$  can diagonalized, then the eigenvectors of  $\mathbf{L}_\mathbf{A}$  form a basis of  $H^r$ . The subset of eigenvectors of  $\mathbf{L}_\mathbf{A}$  with non-zero eigenvalues then form a basis of the image,  $B^r$ , of  $H^r$  under  $\mathbf{L}_\mathbf{A}$ . It follows that the component of  $\mathbf{X}_r$  in  $B^r$  can be expanded in terms of these eigenvectors and  $\mathbf{h}_r$  chosen such that

$$h_{\mathbf{m},i} = X_{\mathbf{m},i} / \Lambda_{\mathbf{m},i}.$$

to ensure the removal of these terms. The component,  $\mathbf{w}_r$ , of  $\mathbf{X}_r$  lying in the complementary subspace,  $G^r$ , of  $B^r$  in  $H^r$  will be unchanged by the transformations  $\mathbf{x} = \mathbf{y} + \mathbf{h}_r(\mathbf{y})$  obtained from  $B^r$ .

Since

$$\mathbf{X}_r(\mathbf{y} + \mathbf{h}_{r+k}(\mathbf{y})) = \mathbf{X}_r(\mathbf{y}) + O(\|\mathbf{y}\|^{r+k+1}), r \geq 2, k = 1, 2, \dots,$$

these terms are not changed by subsequent transformations to remove non-resonant terms of higher order.

The above facts are expressed in

**Theorem 16 (theorem 2.3.1 in [359])** *Given a smooth vector field  $\mathbf{X}(\mathbf{x})$  on  $\mathbb{R}^n$  with  $\mathbf{X}(0) = 0$ , there is a polynomial transformation to new coordinates,  $\mathbf{y}$ , such that the differential equation  $\mathbf{x}' = \mathbf{X}(\mathbf{x})$  takes the form  $\mathbf{y}' = \mathbf{J}\mathbf{y} + \sum_{r=1}^N \mathbf{w}_r(\mathbf{y}) + O(\|\mathbf{y}\|^{N+1})$ , where  $\mathbf{J}$  is the real Jordan form of  $\mathbf{A} = \mathbf{D}\mathbf{X}(0)$  and  $\mathbf{w}_r \in G^r$ , a complementary subspace of  $H^r$  on  $B^r = \mathbf{L}_\mathbf{A}(H^r)$ .*

### 2.2.5.5 Asymptotic behavior

No we develop the technical apparatus to dealing with the notions of “long term” and “observable” behavior of the orbits in the phase space. We consider vector fields (2.2) with  $\mathbf{X}$  of class  $C^r$  ( $r \geq 1$ .) In the following  $\mathbf{g}^\tau(\mathbf{x})$  denotes the flow generated by the vector field (or differential equation)

$$\mathbf{x}'(\tau) = \mathbf{X}(\mathbf{x}(\tau)), \mathbf{x}(\tau) \in \mathbb{R}^n, \quad (2.49)$$

where the prime denote derivative with respect to  $\tau$ .

**Limit Sets** Let us define the concepts of  $\alpha$  and  $\omega$ -sets.

**Definition 14 (definition 8.1.1, [358] p. 104)** *A point  $\mathbf{x}_0 \in \mathbb{R}^n$  is called an  $\omega$ -limit point of  $\mathbf{x} \in \mathbb{R}^n$ , denoted  $\omega(\mathbf{x})$ , if there exists a sequence  $\{\tau_i\}$ ,  $\tau_i \rightarrow \infty$  such that  $\mathbf{g}^{\tau_i}(\mathbf{x}) \rightarrow \mathbf{x}_0$ .  $\alpha$ -limits are defined similarly by taking a sequence  $\{\tau_i\}$ ,  $\tau_i \rightarrow -\infty$ .*

**Definition 15 (definition 8.1.2, [358] p. 105)** *The set of all  $\omega$ -limit points of a flow or map is called a  $\omega$ -limit set. The  $\alpha$ -limit is similarly defined.*

The following result describe some of the basic properties of the  $\alpha$  and  $\omega$ -limit sets of trajectories.

**Proposition 3 (proposition 8.1.3 [358], p. 105)** *Let  $g^\tau(\cdot)$  be a flow generated by a vector field and let  $M$  be a positively invariant compact set for this flow (see definition 3.0.3 p. 28 [358]). Then for  $p \in M$ , we have*

- i)  $\omega(p) \neq \emptyset$
- ii)  $\omega(p)$  is closed
- iii)  $\omega(p)$  is invariant under the flow, i.e.,  $\omega(p)$  is a union of orbits.
- iv)  $\omega(p)$  is connected.

**Proof.** See [358]

- i) Choose a sequence a sequence  $\{\tau_i\}$ ,  $\tau_i \rightarrow \infty$ , and let  $\{p_i = g^{\tau_i}(p)\}$ . Since  $M$  is compact,  $\{p_i\}$  is a convergent subsequence whose limit belongs to  $\omega(p)$ . Then  $\omega(p) \neq \emptyset$ .
- ii) It is sufficient to show that the complement of  $\omega(p)$  is an open set. Let  $q \notin \omega(p)$ . Then, there exists a neighborhood of  $q$ ,  $U(q)$ , which is disjoint to the set of points  $\{g^\tau(p) | \tau > T\}$  for some  $\tau \geq T$ . Then,  $q$  is contained in an open set that do not contain any point in  $\omega(p)$ . Since  $q$  is arbitrary, we obtain the desired result.
- iii) Let  $q \in \omega(p)$  and  $\tilde{q} \in g^s(q)$ . Choose a sequence  $\{\tau_i\}$ ,  $\tau_i \rightarrow +\infty$  when  $i \uparrow +\infty$ . Since  $g^{\tau_i}(p) \rightarrow q$ , then  $g^{\tau_i+s}(p) = g^s(g^{\tau_i}(p))$ , converges to  $\tilde{q}$  as  $i \rightarrow +\infty$ . Thus,  $\tilde{q} \in \omega(p)$  and then,  $\omega(p)$  is invariant.

In this proof we have assumed that  $g^s(\cdot)$  exist for all  $s$ . However, this fact it not so obvious. Let us prove the above statement for  $q \in \omega(p)$ , that is, let us prove that  $g^s(q)$  exist for  $s \in (-\infty, +\infty)$  for all  $q \in \omega(p)$ . It is clear that this is true for  $s \in (0, +\infty)$  since  $M$  is a positively invariant compact set. Then, it is sufficient to prove that it is true for  $s \in (-\infty, 0]$ .

Since  $q \in \omega(p)$ , there exists a time sequence  $\{\tau_i\}$ ,  $\tau_i \rightarrow +\infty$  as  $i \uparrow +\infty$ , such that  $g^{\tau_i}(p) \rightarrow q$  as  $i \rightarrow +\infty$ . Let us sort the sequence such that  $\tau_1 < \tau_2 < \dots < \tau_n < \dots$ , and consider  $g^s(g^{\tau_i}(p))$ . Following proposition 1, the above composition is well-defined for  $s \in [-\tau_i, 0]$ . Taking the limit as  $i \rightarrow +\infty$ , using the continuity of the flow and the fact that  $g^{\tau_i}(p) \rightarrow q$  as  $i \rightarrow +\infty$ , we see that  $g^s(q)$  exist for  $s \in (-\infty, 0]$ .

1. The proof is by contradiction. Suppose that  $\omega(\mathbf{p})$  is not connected. Then we can choose open sets  $V_1, V_2$  such that  $\omega(\mathbf{p}) \in V_1 \cup V_2$ ,  $\omega(\mathbf{p}) \cap V_1 \neq \emptyset$ ,  $\omega(\mathbf{p}) \cap V_2 \neq \emptyset$ , and  $\bar{V}_1 \cap \bar{V}_2 = \emptyset$ .

The orbit  $O(\mathbf{p})$  accumulates both in points of  $V_1$  and in points of  $V_2$ ; thus, for a given  $T > 0$ , exists  $\tau > T$  such that  $\mathbf{g}^\tau(\mathbf{p}) \in M \setminus (V_1 \cup V_2) = K$ , a compact set. Then, we can find a sequence  $\tau_n \rightarrow +\infty$  as  $n \uparrow +\infty$  with  $\mathbf{g}^{\tau_n}(\mathbf{p}) \in K$ . Passing to a subsequence, if necessary ( $K$  is compact), we have  $\mathbf{g}^{\tau_n}(\mathbf{p}) \rightarrow \mathbf{q}$ ,  $\mathbf{q} \in K$ . This implies that  $\mathbf{q} \in V_1 \cup V_2$ . However, our construction indicates that  $\mathbf{q} \in \omega(\mathbf{p})$ . This is a contradiction with the hypothesis  $\bar{V}_1 \cap \bar{V}_2 = \emptyset$ .

It can be proved an analogous result for  $\alpha$ -limit sets if the hypothesis of proposition 3 holds for a flow in reversed time.

**Attracting sets, attractors and basin of attraction** Now we want to develop the idea of an attractor.

**Definition 16 (Attracting set for flows, definition 8.2.1, [358])** *A closed invariant set  $A \subset \mathbb{R}^n$ , is called an attracting set, if there exists a neighborhood  $U$  of  $A$  such that  $\forall \tau \geq 0, \mathbf{g}^\tau(U) \subset U$  and  $\cap_{\tau > 0} \mathbf{g}^\tau(U) = A$ .*

**Definition 17 (Trapping region, definition 8.2.2, [358])** *The open set  $U$  in the definition 16 is often referred to as a trapping region.*

A similar definition is given in

**Definition 18** *Given the vector field (2.2) in  $\mathbb{R}^n$ , with flow  $\{\mathbf{g}^\tau\}$ , a subset  $S \subseteq \mathbb{R}^n$  is said to be a trapping set of the DE if it satisfies:*

1.  $S$  is a closed and bounded set,
2.  $\mathbf{y} \in S$  implies that  $\mathbf{g}^\tau(\mathbf{y}) \in S$  for all  $\tau \geq 0$ .

To find a Lyapunov function is equivalent to finding a trapping region. By theorem 1 it follows that all solution starting in a trapping region exists for all positive times. This is useful in noncompact phase spaces such as  $\mathbb{R}^2$  for proving existence on semi-infinite time intervals. In the continuous time case, one “test” whether or not a region is a candidate to be a trapping region by evaluating the vector field on the boundary of the region in question. If, on the boundary of the region, the vector field is pointing toward the interior of the region, or it is tangent to the boundary, then, the given section is a trapping region. However, in order to the test be carried out, one needs that the boundary of the region must be, at least,  $C^1$ .

Another idea related with trapping regions is that of absorbing set.

**Definition 19 (Absorbing set for flows, definition 8.2.3, [358])** A positive invariant compact subset  $B \subset \mathbb{R}^n$  is called an absorbing set if there exists a bounded subset of  $\mathbb{R}^n$ ,  $U$ , with  $U \supset B$ , and  $\tau_U$  such that  $\mathbf{g}^\tau(U) \subset B$ ,  $\forall \tau \geq \tau_U$ .

If we have an attracting set it is natural to ask which points in phase space approach the attracting set asymptotically.

**Definition 20 (Basin of attraction for flows, definition 8.2.4, [358])** The domain or basin of attraction of an attracting set  $A$  is given by

$$\bigcup_{\tau \leq 0} \mathbf{g}^\tau(U),$$

where  $U$  is any open set satisfying definition 16.

**Note.** The basin of attraction is independent of the choice of the open set  $U$ , provided that  $U$  satisfies definition 16.

An attracting set can contain several sinks (“attractors”) and almost all the points in the phase space will eventually end up near one of these sinks (see example 8.2.2, [358]). Therefore, if we are interested in describing where most points in the phase space ultimately go, the idea is an attracting set is not quite precise enough. It is necessary to incorporate into the definition the notion that it is not only a collection of distinct attractors, but rather, that all points in the attracting set eventually come arbitrarily close to every other point in the attracting set under the evolution of the flow.

This idea can be rigorously stated as

**Definition 21 (Topological transitivity, definition 8.2.5, [358])** A closed invariant set  $A$  is said to be topologically transitive if, for any two open sets  $U, V \subset A$  exists  $\tau \in \mathbb{R}$  such that  $\mathbf{g}^\tau(U) \cap V \neq \emptyset$ .

**Definition 22 (Attractor, definition 8.2.6, [358])** An attractor is a topologically transitive attracting set.

**LaSalle Invariance Principle, Monotone Functions and Monotonicity Principle.** In this section we describe an application of the invariance of  $\omega$ -limit sets of a trajectory that is very useful for the stability analysis. It is referred as the LaSalle Invariance Principle (see [361]; theorem 8.3.1 in [358], p. 111).

Given the autonomous vector field (2.2) where  $\mathbf{X}$  is of class  $C^r$  ( $r \geq 1$ ). Let  $\mathcal{M} \subset \mathbb{R}^n$  be a positively invariant compact set under the flow,  $\mathbf{g}^\tau(\cdot)$ , generated by this vector field, which is the closure of some open set (so that it has a nonempty interior) and whose boundary is (at least)  $C^1$ . Therefore,  $\mathcal{M}$  is a trapping region. Let  $V(\mathbf{x})$  a Lyapunov function on  $\mathcal{M}$ . By this we mean that  $V'(\mathbf{x}) \equiv \nabla V(\mathbf{x}) \cdot \mathbf{X}(\mathbf{x}) \leq 0$  on  $\mathcal{M}$  (here we use

the notion of Lyapunov function not as in the context of theorem 6, as a local notion in a neighborhood of a fixed point, but in a broader global sense). Consider the two sets

$$E \equiv \{\mathbf{x} \in \mathcal{M} | V'(\mathbf{x}) = 0\}$$

and

$$M \equiv \left\{ \begin{array}{l} \text{The union of all the trajectories that start in } E \\ \text{and remain in } E \text{ for all } \tau > 0 \end{array} \right\}.$$

$M$  is a “positively invariant part” of  $E$ . Now we can state the LaSalle Invariance Principle.

**Theorem 17 (LaSalle Invariance Principle)** *For all  $\mathbf{x} \in \mathcal{M}$ ,  $\mathbf{g}^\tau(\mathbf{x}) \rightarrow M$  ([361]; theorem 8.3.1 in [358], p. 111).*

**Proof.** (See [358], p. 111) First, let us prove that  $V$  is a constant  $\chi$  on  $\omega(\mathbf{x})$ . Suppose that  $\bar{\mathbf{x}} \in \omega(\mathbf{x})$  and let  $\chi = V(\bar{\mathbf{x}})$ , then,  $\chi$  is greatest lower bound of the set  $\{V(\mathbf{g}^\tau(\mathbf{x})) | \tau \geq 0\}$ . This follows from the fact that  $V$  decreases along trajectories (hence  $V(\mathbf{g}^{\tau_i}(\mathbf{x})) \geq V(\mathbf{g}^\tau(\mathbf{x})) \geq V(\mathbf{g}^{\tau_{i+1}}(\mathbf{x}))$  for  $\tau_i \leq \tau \leq \tau_{i+1}$ ) and by the continuity of  $V$ . From proposition 3, the omega limit set of a trajectory is invariant, hence  $\mathbf{g}^\tau(\bar{\mathbf{x}})$  is also an omega limit point of  $\mathbf{g}^\tau(\mathbf{x})$ . Then, since  $\chi$  is the greatest lower bound of the set  $\{V(\mathbf{g}^\tau(\mathbf{x})) | \tau \geq 0\}$ ,  $V(\mathbf{g}^\tau(\bar{\mathbf{x}})) = \chi$ . From this it follows that  $V' = 0$  on  $\omega(\mathbf{x})$ . Then, by the definition of  $E$ ,  $\omega(\mathbf{x}) \subset E$ . Since  $\omega(\mathbf{x})$  is invariant (proposition 3), it follows by the definition of  $M$  that  $\omega(\mathbf{x}) \subset M$ . Therefore  $\mathbf{g}^\tau(\mathbf{x}) \rightarrow M$  as  $\tau \rightarrow +\infty$ .

LaSalle Invariance Principle can be generalized considerably, for instance to theorem 18. This extension requires the introduction of the concept of monotonic function for the flow.

**Definition 23 (definition 4.8 [1], p. 93)** *Let  $\mathbf{g}^\tau(\mathbf{x})$  be a flow on  $\mathbb{R}^n$ , let  $S$  be an invariant set of  $\mathbf{g}^\tau(\mathbf{x})$  and let  $Z : S \rightarrow \mathbb{R}$  be a continuous function.  $Z$  is monotonic decreasing (increasing) function for the flow  $\mathbf{g}^\tau(\mathbf{x})$  means that for all  $\mathbf{x} \in S$ ,  $Z(\mathbf{g}^\tau(\mathbf{x}))$  is a monotonic decreasing (increasing) function of  $\tau$ .*

**Theorem 18 (Monotonicity Principle)** *Let  $\mathbf{g}^\tau(\mathbf{x})$  be a flow on  $\mathbb{R}^n$  with  $S$  an invariant set. Let  $Z : S \rightarrow \mathbb{R}$  be a  $C^1(\mathbb{R}^n)$  function whose range is the interval  $(a, b)$  where  $a \in \mathbb{R} \cup \{-\infty\}$ ,  $b \in \mathbb{R} \cup \{+\infty\}$ , and  $a < b$ . If  $Z$  is decreasing on orbits in  $S$ , then for all  $\mathbf{x} \in S$ ,  $\omega(\mathbf{x}) \subset \{\mathbf{s} \in \bar{S} \setminus S | \lim_{\mathbf{y} \rightarrow \mathbf{s}} Z(\mathbf{y}) \neq b\}$  and  $\alpha(\mathbf{x}) \subset \{\mathbf{s} \in \bar{S} \setminus S | \lim_{\mathbf{y} \rightarrow \mathbf{s}} Z(\mathbf{y}) \neq a\}$ .*

**Results for Planar Systems** The Poincaré-Bendixon theorem gives us a complete determination of the asymptotic behavior of a large class of flows on the plane, cylinder, and two-sphere. It is remarkable in that it assumes not detailed information about the vector field, only uniqueness of the solutions, properties of  $\omega$ -limit sets, and some properties of the geometry of the underlying phase plane.

**Theorem 19 (Poincaré-Bendixon Theorem)** *Let  $\mathcal{M}$  be a positively invariant set for the vector field (2.49) on  $\mathbb{R}^2$  (with  $\mathbf{X} \in C^2$ ), containing at most a finite number of singular points (i.e., no non-isolated singular points). Let  $\mathbf{p} \in \mathcal{M}$ , and consider  $\omega(\mathbf{p})$ . Then one of the following possibilities holds:*

1.  $\omega(\mathbf{p})$  is a singular point.
2.  $\omega(\mathbf{p})$  is a closed (periodic) orbit.
3.  $\omega(\mathbf{p})$  is the union of singular points and heteroclinic or homoclinic orbits. That is  $\omega(\mathbf{p})$  consists of a finite number of singular points  $\mathbf{p}_1, \dots, \mathbf{p}_n$  and orbits  $\gamma$  with  $\alpha(\gamma) = \mathbf{p}_i$  and  $\omega(\gamma) = \mathbf{p}_j$ ,  $i, j \in \{1 \dots n\}$ .

**Theorem 20 (Corollary of Poincaré-Bendixon Theorem 19, [2], p. 22)** *Let be  $K$  a positively invariant set for the vector field (2.49) on  $\mathbb{R}^2$  (with  $\mathbf{X} \in C^1$ ). If  $K$  is a bounded and closed set, the  $K$  contains either a closed (periodic) orbit, or a singular point.*

**Theorem 21 (Dulac's Criterion)** *If  $D \subseteq \mathbb{R}^2$  is a simply connected open set and  $B$  is a Dulac's function on  $D$ , then the differential equation (2.49) on  $\mathbb{R}^2$ , with  $\mathbf{X} \in C^1$  has no periodic (or closed) orbit which is contained in  $D$ .*

## 2.3 Procedure for analyzing cosmological dynamical systems

Given a cosmological dynamical system determined by the differential equation

$$\frac{dy}{d\tau} = f(y), y \in \mathbb{R}^n, \quad (2.50)$$

$$g(y) = 0, \quad (2.51)$$

the standard procedure to analyze the properties of the flow generated by (2.50) subject to the constraint(s) (2.51) (see, for example, the reference [1]) is the following:

1. Determine whether the state space, as defined by (2.51), is compact.
2. Identify the lower-dimensional invariant sets, which contains the orbits of more special classes of models with additional symmetries.
3. Find all the singular points and analyze their local stability. Where possible identify the stable and unstable manifolds of the singular points, which may coincide with some of the invariant sets in point (2).



4. Find Dulac's functions or monotone functions in various invariant sets where possible.
5. Investigate any bifurcation that occur as the equation of state parameter  $\gamma$  (or any other parameters) varies. The bifurcations are associated with changes in the local stability of the singular points.
6. Having all the information in the points (1)-(5) one can hope to formulate precise conjectures about the asymptotic evolution, by identifying the past and the future attractors. The past attractor will describe the evolution of a typical universe near the initial singularity while the future attractor will play the same role at late times. The monotone functions in point (4) above, in conjunction with theorems of dynamical systems theory, may enable some of the conjectures to be proved.
7. Knowing the stable and unstable manifolds of the singular points it is possible to construct all possible heteroclinic sequences that join the past attractor, thereby gaining insight into the intermediate evolution of cosmological models.



# Chapter 3

## Non-minimally Coupled Dark Energy Models

In this chapter we investigate, from the dynamical systems viewpoint flat FRW models in the conformal (Einstein) frame of scalar-tensor gravity theories including  $f(R)$  theories through conformal transformation. Particularly we are interested in investigating the stability of the de Sitter solution in this framework that give an answer to the accelerating expansion. Also we investigate the stability of scaling solutions. Scaling late-time attractor solutions provide a hint for solving or alleviating the Coincidence Problem.

### 3.1 Introduction

Current astrophysical observations suggest that the universe is permeated by an exotic form of matter called Dark Energy that is driving the current accelerated expansion [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] which can be modelled as a self-interacting scalar field.

Scalar fields, and theories including it such as Scalar-tensor theories (STT) of gravity [208, 209, 210, 211, 212, 213, 214] can be supported by fundamental physical theories like superstring theory [215]. Quintessential DE models [27, 163, 162], for instance, are described by an ordinary scalar field minimally coupled to gravity. A particular choice of the scalar field self-interacting potentials can drive the past and current accelerated expansion. The natural generalizations to quintessence models evolving independently from the background matter are models that exhibit non-minimal coupling between both components. The effective interaction dark energy-dark matter appears when we apply conformal transformations <sup>1</sup> to the STTs. Several physical theories which predict the presence of a scalar field coupled to matter. For example, in string theory the dilaton field is generally coupled to matter [375]. Nonminimally coupling occurs also in STT of

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<sup>1</sup> See the reference [235] for applications of conformal transformations in both relativity and cosmology.

gravity [376, 377], in HOG theories [80] and in models of chameleon gravity [378]. Coupled quintessence was investigated also in [193, 379, 380] by using dynamical systems techniques.

The cosmological dynamics of scalar-tensor gravity have been investigated in [112, 381]. Phenomenological coupling functions were studied for instance in [382] which can describe either the decay of dark matter into radiation, the decay of the curvaton field into radiation or the decay of dark matter into dark energy (see section III of [382] for more information and for useful references). In the reference [381], the authors construct a family of viable scalar-tensor models of dark energy (which includes pure  $F(R)$  theories and quintessence). By investigating a phase space the authors obtain that the model possesses a phase of late-time acceleration preceded by a standard matter era, while at the same time satisfying the local gravity constraints (LGC). In [383] it is studied a scalar field responsible for both the early and the late time inflationary expansion in the context of brane cosmology.

In the inflationary universe scenarios (mainly based on GRT) matter is modelled, usually, as a scalar field,  $\phi$ , with potential  $V(\phi)$ , which must meet the requirements necessary to lead to the early-time accelerating expansion [384, 385, 386, 387, 388]. If the potential is constant, i.e., if  $V(\phi) = V_0$ , space-time is de Sitter and expansion is exponential. If the potential is exponential, i.e.,  $V(\phi) = V_0 \exp[-\lambda\phi]$ , we get an inflationary powerlaw solution [389, 390]. Several gravity theories consider multiple scalar fields with exponential potential, particularly assisted inflation scenarios [391, 392, 393, 394, 395], quintom dark energy paradigm [56, 57, 60, 357] and others. Also, have been considered positive and negative exponential potentials [396], single exponential and double exponential [397, 398, 399, 400], etc. Other generalizations with multiple scalar fields are available [401, 402].

The dynamical behavior of space-times based on GRT is so far known for a large variety of models with scalar fields with non-negative potential [241, 242, 403, 404, 405, 406]. In reference [406], have been extended many of the results obtained in [404] considering arbitrary potentials. In [403] it has been shown that for a large class of FRW cosmologies with scalar fields with arbitrary potential, the past attractor is a family of solutions in one-to-one correspondence with exactly integrable cosmologies with a massless scalar field. This result has been extended somewhat in [136] to FRW cosmologies based on STTs. In this reference was investigated a general model of coupled dark energy with arbitrary potential  $V$  and coupling function  $\chi$ . It was proved there, by using dynamical systems techniques that if the potential and the coupling function are sufficiently smooth functions; the scalar field almost always diverges into the past. Under some regularity conditions for the potential and for the coupling function in that limit, it was constructed a dynamical system well suited to investigate the dynamics near the initial singularity. The singular points therein were investigated and the cosmological solutions associated to them were

characterized. There was presented asymptotic expansions for the cosmological solutions near the initial space-time singularity, which extend previous results of [403]. On the other hand, in [407] it was investigated flat and negatively curved Friedmann-Robertson-Walker (FRW) models with a perfect fluid matter source and a scalar field arising in the conformal frame of  $F(R)$  theories nonminimally coupled to matter. It was proved there that, for a general class of potentials  $V$ , the equilibrium corresponding to non-negative local minima for  $V$  are asymptotically stable, as well as horizontal asymptotes approached from above by  $V$ . For a nondegenerated minimum of the potential with zero critical value they prove in detail that if  $\gamma > 1$ , then there is a transfer of energy from the fluid to the scalar field and the later eventually dominates in a generic way. As we will see in next sections the results in [407] and in [136] can be obtained by investigating a general class of models containing both STTs and  $F(R)$  gravity.

## 3.2 The Field Equations

In this section we consider a phenomenological model inspired in the action (1.77) for FRW space-times with flat spatial slices, modelled by the metric:

$$ds^2 = -dt^2 + a(t)^2 (dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)). \quad (3.1)$$

We use a system of units in which  $8\pi G = c = \hbar = 1$ . We assume that the energy-momentum tensor (1.78) is in the form of a perfect fluid

$$T_{\beta}^{\alpha} = \text{diag}(-\rho, p, p, p),$$

where  $\rho$  and  $p$  are respectively the isotropic energy density and the isotropic pressure (consistently with FRW metric, pressure is necessarily isotropic [160]). For simplicity we will assume a barotropic equation of state  $p = (\gamma - 1)\rho$ . Also we consider a quintessence scalar field,  $\phi$ , interacting in the action with the background pressureless dark matter fluid. As in [200], here the baryons (a subdominant component at present, but important in the past of the cosmic evolution) are included in the background of dark matter. In fact, there is the possibility of a universal coupling of dark energy to all sorts of matter, including baryons (but excluding radiation) [408]. We include radiation in the cosmic budget since it is an important matter source in the early universe. However, in some cases of interest we set  $\rho_r = 0$ .

The cosmological equations for flat FRW models with a scalar field coupled to matter an including also radiation are given by

$$\dot{H} = -\frac{1}{2} \left( \gamma\rho + \frac{4}{3}\rho_r + \dot{\phi}^2 \right), \quad (3.2)$$

$$\dot{\rho} = -3\gamma H\rho - \frac{1}{2}(4-3\gamma)\rho\dot{\phi} \frac{d \ln \chi(\phi)}{d\phi}, \quad (3.3)$$

$$\dot{\rho}_r = -4H\rho_r, \quad (3.4)$$

$$\ddot{\phi} = -3H\dot{\phi} - \frac{dV(\phi)}{d\phi} + \frac{1}{2}(4-3\gamma)\rho \frac{d \ln \chi(\phi)}{d\phi}, \quad (3.5)$$

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho + \rho_r. \quad (3.6)$$

where  $a$  is the scale factor,  $H = \dot{a}/a$  is the Hubble parameter,  $\rho$  denotes the energy density of barotropic matter,  $\rho_r$  denotes de energy density of radiation,  $\phi$  denotes the scalar field and  $V(\phi)$  and  $\chi(\phi)$  are, respectively, the potential and coupling functions.

To maintain the analysis as general as possible, we will not specify the functional forms of the potential and the coupling function from the beginning. Instead we consider the general hypothesis  $V(\phi) \in C^3, V(\phi) > 0, \chi(\phi) \in C^3$  and  $\chi(\phi) > 0$ . We impose they in order to obtain dynamical systems of class  $C^2$ . However, to derive some of our results we will relax some of this hypothesis, or consider further assumptions (they will be clearly stated when applicable). We consider also  $\rho \geq 0$  and  $0 < \gamma < 2, \gamma \neq \frac{4}{3}$ . These hypotheses for background matter are the usual. We assume  $\gamma \neq \frac{4}{3}$  to exclude the possibility that the background matter behaves as radiation. The energy momentum tensor for radiation ( $\gamma = 4/3$ ) is traceless, so it is automatically decoupled from a scalar field non-minimally coupled to dark matter in the Einstein frame. Radiation source is added by hand in order to model also the cosmological epoch when barotropic matter and radiation coexisted since we want to investigate the possible scaling solutions in the radiation regime. We neglect ordinary (uncoupled) barotropic matter.

The strength of the coupling between the perfect fluid and the scalar field is defined by

$$\delta \equiv -\frac{1}{2}(4-3\gamma)\rho\dot{\phi} \frac{d \ln \chi(\phi)}{d\phi}.$$

One of the first papers to take seriously the possibility of interaction in scalar field cosmologies, from the dynamical system perspective, was [380]. In this paper we can find a review on the subject. There it was investigated the interaction terms (in the flat FRW geometry)  $\delta = -\alpha\dot{\phi}\rho$  and  $\delta = \alpha\rho H$ , where  $\alpha$  is a constant,  $\phi$  is the scalar field,  $\rho$  is the energy density of background matter and  $H$  stands for the Hubble parameter. The first choice corresponds to an exponential coupling function  $\chi(\phi) = \chi_0 \exp(2\alpha\phi/(4-3\gamma))$ . The second case corresponds to the choice  $\chi = \chi_0 a^{-2\alpha/(4-3\gamma)}$  (and then,  $\rho \propto a^{\alpha-3\gamma}$ , where  $a$  denotes the scale factor of the Universe (recall that the former derivations are only valid in a flat FRW model). Other phenomenological coupling functions were stud-

ied elsewhere. We want to draw the attention of the reader to a physically well motivated approach to the coupling function in [382]. In that paper it was investigated a coupling term of the form  $\delta = -\alpha\rho$ , where  $\alpha$  is a constant ( $\Gamma$  in their notation). As commented in that reference, if  $\alpha > 0$ , the model can describe either the decay of dark matter into radiation, the decay of the curvaton field into radiation or the decay of dark matter into dark energy (see section III of [382] for more information and for useful references). In the reference [381], the authors construct a family of viable scalar-tensor models of dark energy (which includes pure  $f(R)$  theories and quintessence). They consider a coupling between the scalar field and the non-relativistic matter in the Einstein frame of the type -in our notation-  $\chi(\phi) = e^{-2Q\phi}$ , with  $Q$  constant. By investigating a phase space the authors obtain that the model posses a phase of late-time acceleration preceded by a standard matter era, while at the same time satisfying the local gravity constraints (LGC). In fact, by studying the evolution of matter density perturbations and employing them, the authors place bounds on the coupling of the order  $|Q| < 2.5 \times 10^{-3}$  (for the massless case). By a chameleon mechanism the authors show that these models can be made compatible with LGC even when  $|Q|$  is of the order of unity if the scalar-field potential is chosen to have a sufficiently large mass in the high-curvature regions.

In order to classify the global behavior of the solutions of (1.77) it is required a detailed knowledge of the global form of the scalar field potential (and of the coupling function  $\chi$ ). However, up to the present, there exist no consensus about the specific functional form of  $V(\phi)$  (and of  $\chi(\phi)$ ) for make calculations. As a consequence it would be of interest classify the dynamical behavior of solutions without specifying the functional form of the potential function (and of the coupling function). In the literature of General Relativity (GR) relevant attempts have been made in this more general direction [136, 241, 243, 244, 242].

In this investigation we want to study, from the dynamical systems point of view, a phenomenological model inspired in a STT with action (1.77) where the matter and the (quintessence) scalar field are coupled in the action (1.77) through the scalar tensor metric  $\chi(\phi)^{-1}g_{\alpha\beta}$  [192]. We consider arbitrary functional form for self-interaction potential and the coupling function for the scalar field,  $\phi$ . When we take the conformal transformation allowing writing the action in the JF as in (1.79) the coupling function  $\chi$  should be interpreted as the dilation (BD) field and the corresponding  $\omega(\chi)$  as the varying BD parameter.

The aim of the chapter is to extent several results in [136, 403, 404, 407] to the more realistic situation when radiation is included in the cosmic budget (particularly for investigating the early-time dynamics). We will focus mainly in a particular era of the universe where matter and radiation coexisted. Otherwise the inclusion of radiation complicates the study in an unnecessary manner, since, assuming a perfect barotropic fluid with an arbitrary barotropic index  $\gamma$ , for  $\gamma = 4/3$ , this matter source corresponds to radiation.

Thus we will consider both ordinary matter described by a perfect fluid with equation of state  $p = (\gamma - 1)\rho$  (coupled to a scalar field) and radiation with energy density  $\rho_r$ . We are interested in investigate all possible scaling solutions in this regime. Although we are mainly interest in describing the early time dynamics of our model, for completeness we will focus also in the late-time dynamics. As in [407] we obtain for flat FRW models sufficient conditions under the potential, to establish the asymptotic stability of the non-negative local minima for  $V(\phi)$ . Center manifold theory is employed to analyze the stability solutions associated to the local degenerated minimum and the inflection points of the potential. We prove for arbitrary potentials and arbitrary coupling functions  $\chi(\phi)$ , of appropriate differentiable class, that the scalar field almost always diverges into the past generalizing the results in [403, 136]). It is designed a dynamical system adequate to studying the stability of the singular points in the limit  $|\phi| \rightarrow \infty$ . We obtain there: radiation-dominated cosmological solutions; power-law scalar-field dominated inflationary cosmological solutions; matter-kinetic-potential scaling solutions and radiation-kinetic-potential scaling solutions. It is discussed, by means of several worked examples, the link between our results and the results obtained for specific  $F(R)$  frameworks by using appropriated conformal transformations. We illustrated both analytically and numerically our principal results. Particularly, we investigate the important examples of higher order gravity theories  $F(R) = R + \alpha R^2$  (quadratic gravity) and  $F(R) = R^n$ . In the case of quadratic gravity we prove, by an explicit computation of the center manifold, that the singular point corresponding to *de Sitter* solution is locally unstable (saddle point). This result complements the result of the proposition discussed in [405] p. 5, where it was proved the local asymptotic instability of the de Sitter universe for positively curved FRW models with a perfect fluid matter source and a scalar field which arises in the conformal frame of the  $R + \alpha R^2$  theory. Finally, we investigate a general class of potentials containing the cases investigated in [409, 410]. In order to provide a numerical elaboration for our analytical results for this class of models, we re-examine the model with power-law coupling and Albrecht-Skordis potential  $V(\phi) = e^{-\mu\phi}(A + (\phi - B)^2)$  investigated in [136] in presence of radiation. Also, we investigate in detail the invariant set  $\rho_r = 0$  obtaining several results for the early- and late-time universe. In particular we formalize and prove two singularity theorems.

### 3.3 Late Time Behavior

In the following, we study the late time behavior of solutions of (3.2), (3.3), (3.4), (3.5), which are expanding at some initial time, i. e.,  $H(0) > 0$ . The state vector of the system



is  $(\phi, \dot{\phi}, \rho, \rho_r, H)$ . Defining  $y := \dot{\phi}$ , we rewrite the autonomous system as

$$\dot{\phi} = y, \quad (3.7)$$

$$\dot{y} = -3Hy - \frac{dV(\phi)}{d\phi} + \frac{1}{2}(4 - 3\gamma)\rho \frac{d \ln \chi(\phi)}{d\phi}, \quad (3.8)$$

$$\dot{\rho} = -3\gamma H\rho - \frac{1}{2}(4 - 3\gamma)\rho y \frac{d \ln \chi(\phi)}{d\phi}, \quad (3.9)$$

$$\dot{\rho}_r = -4H\rho_r, \quad (3.10)$$

$$\dot{H} = -\frac{1}{2} \left( \gamma\rho + \frac{4}{3}\rho_r + y^2 \right), \quad (3.11)$$

subject to the constraint

$$3H^2 = \frac{1}{2}y^2 + V(\phi) + \rho + \rho_r. \quad (3.12)$$

**Remark 1** Using standard arguments of ordinary differential equations theory, follows from equations (3.9) and (3.10) that the signs of  $\rho$  and  $\rho_r$ , respectively, are invariant. This means that if  $\rho > 0$  and  $\rho_r > 0$  for some initial time  $t_0$ , then  $\rho(t) > 0$ , and  $\rho_r(t) > 0$  throughout the solution. From (3.11) and (3.12) and only if additional conditions are assumed, for example  $V(\phi) \geq 0$  and  $V(\phi_*) = 0$  for some  $\phi_*$ , follows that the sign of  $H$  is invariant. From (3.10) and (3.11) follows that  $\rho_r$  and  $H$  decreases. Also, defining  $\epsilon = \frac{1}{2}y^2 + V(\phi)$ , follows from (3.8)-(3.9) that

$$\dot{\epsilon} + \dot{\rho} = -3H(y^2 + \gamma\rho). \quad (3.13)$$

Thus, the total energy density contained in the dark sector is decreasing.

First we study the simpler case were  $\rho_r = 0$ , then, we investigate the general case.

For  $\rho_r = 0$ , equations (3.7)-(3.11) becomes

$$\dot{H} = -\frac{1}{2}(\gamma\rho + y^2), \quad (3.14)$$

$$\dot{\rho} = -3\gamma H\rho - \frac{1}{2}(4 - 3\gamma)\rho y \frac{d \ln \chi(\phi)}{d\phi}, \quad (3.15)$$

$$\dot{y} = -3Hy - \frac{dV(\phi)}{d\phi} + \frac{1}{2}(4 - 3\gamma)\rho \frac{d \ln \chi(\phi)}{d\phi}, \quad (3.16)$$

$$\dot{\phi} = y, \quad (3.17)$$

defining a dynamical system in the phase space

$$\Omega = \{(H, \rho, y, \phi) \in \mathbb{R}^4 | 3H^2 = \frac{1}{2}y^2 + V(\phi) + \rho\}. \quad (3.18)$$

First, let us consider a potential function with a local minimum  $V(0) = 0$ . With this hypothesis the point  $(0, 0, 0, 0)$  is a singular point of (3.14)-(3.17). This fact can be used

to show that an initially expanding universe ( $H > 0$ ) should expand forever. Indeed, the set  $\{(H, \rho, y, \phi) \in \Omega | H = 0\}$  is invariant under the flow of (3.14)-(3.17). Besides, the sign of  $H$  is invariant. If the sign of  $H$  changes, a trajectory with  $H(0) > 0$  can pass through  $(0, 0, 0, 0)$ , violating the existence and uniqueness theorem 1.

The proposition 2 of [404] can be generalized to this context as follows.

**Proposition 4** *Suppose that  $V \geq 0$  and  $V(\phi) = 0 \Leftrightarrow \phi = 0$ . Let  $A$  such that  $V$  bounded in  $A$  implies  $V'(\phi)$  is bounded in  $A$ . If there exists a constant  $K$ ,  $K \neq 0$  such that*

$$\chi'(\phi)/\chi(\phi) \leq 2K/(2 - \gamma)(4 - 3\gamma).$$

Then,

$$\lim_{t \rightarrow \infty} \rho = 0 = \lim_{t \rightarrow \infty} y.$$

**Proof.** Consider the trajectory passing through an arbitrary point  $(H, \rho, y, \phi) \in \Omega$  with  $H > 0$  at  $t = t_0$ . Since  $H$  is positive and decreasing we have that  $\lim_{t \rightarrow \infty} H(t)$  exists and it is a nonnegative number  $\eta$ ; besides,  $H(t) \leq H(t_0)$  for all  $t \geq t_0$ . The, from the restriction (3.18) follows that each term  $\rho$ ,  $1/2y^2$ , and  $V(\phi)$  is bounded by  $3H(t_0)^2$  for all  $t \geq t_0$ .

Let  $A = \{\phi : V(\phi) \leq 3H(t_0)^2\}$ . Then, the trajectory is such that  $\phi$  remains in the interior of  $A$ .

From equation (3.14) we have that

$$-\int_{t_0}^t \left( \frac{1}{2}y^2 + \frac{\gamma}{2}\rho \right) dt = H(t) - H(t_0)$$

and taking the limit as  $t \rightarrow \infty$ , we obtain

$$\frac{1}{2} \int_{t_0}^{\infty} (y^2 + \gamma\rho) dt = H(t_0) - \eta$$

besides,

$$\int_{t_0}^{\infty} (y^2 + \gamma\rho) dt < \infty. \quad (3.19)$$

Taking the time derivative of  $f(t) = y^2 + \gamma\rho$  and making use of the hypothesis for  $\chi(\phi)$  we obtain

$$\frac{d}{dt} (y^2 + \gamma\rho) \leq y(-2V'(\phi) + K\rho).$$

As we have seen,  $y$  and  $\rho$  are bounded, and by the hypothesis for  $V(\phi)$ ,  $V'(\phi)$  is bounded. From this facts follow that the time derivative of  $f$  is bounded. Since  $f$  is a nonnegative

function, the convergence of  $\int_{t_0}^{\infty} f(t)dt$  implies  $\lim_{t \rightarrow \infty} f(t) = 0$ . Hence, we have that

$$\lim_{t \rightarrow \infty} \rho = 0 = \lim_{t \rightarrow \infty} y.$$

■

The hypotheses in 4 are satisfied by a large class of potentials as commented in [404] (this result is extensible to the case of non-minimal coupling), and by a large class of coupling functions including functions dominated by exponential ones.

Under the same hypothesis of proposition 4, we can generalize the proposition 3 in [404].

**Proposition 5** *Suppose that  $V'(\phi) > 0$  for  $\phi > 0$  and  $V'(\phi) < 0$  for  $\phi < 0$ . Then, under the same hypotheses as in proposition 4,  $\lim_{t \rightarrow \infty} \phi$  exists and is equal to  $+\infty$ , 0 or  $-\infty$ .*

**Proof.** Using the same argument as in proposition 4,  $\exists \lim_{t \rightarrow \infty} H(t) = \eta$ . If  $\eta = 0$ , then by the restriction (3.18) we obtain  $\lim_{t \rightarrow \infty} V(\phi(t)) = 0$ . Since  $V$  is continuous and  $V(\phi) = 0 \Leftrightarrow \phi = 0$  this implies that  $\lim_{t \rightarrow \infty} \phi(t) = 0$ .

Suppose that  $\eta > 0$ . From (3.18) we obtain that  $\lim_{t \rightarrow \infty} V(\phi(t)) = 3\eta^2$ . Therefore, exists  $t'$  such that  $V(\phi) > 3\eta^2/2$  for all  $t > t'$ . From this fact follows that  $\phi$  cannot be zero for some  $t > t'$  because  $\phi = 0 \Leftrightarrow V(\phi) = 0$ . Then, the sign of  $\phi$  is invariant for all  $t > t'$ .

Suppose that  $\phi$  is positive for all  $t > t'$ . Since  $V$  is an increasing function of  $\phi$  in  $(0, +\infty)$ , we have that  $\lim_{t \rightarrow \infty} V(\phi(t)) = 3\eta^2 \leq \lim_{\phi \rightarrow \infty} V(\phi)$ . By the continuity and monotony of  $V$  it is obvious that the equality holds if, and only if,  $\lim_{t \rightarrow \infty} \phi(t) = +\infty$ .

If  $\lim_{t \rightarrow \infty} V(\phi(t)) < \lim_{\phi \rightarrow \infty} V(\phi)$ , then there exists  $\bar{\phi} \geq 0$  such that

$$\lim_{t \rightarrow \infty} V(\phi(t)) = V(\bar{\phi}).$$

Since  $V$  is continuous and strictly increasing we have that

$$\lim_{t \rightarrow \infty} \phi = \bar{\phi}.$$

By proposition 4,  $\lim_{t \rightarrow \infty} \rho(t) = 0 = \lim_{t \rightarrow \infty} y(t)$ . Besides,  $H$  and  $\chi'(\phi)/\chi(\phi)$  are bounded. Therefore, taking the limit as  $t \rightarrow \infty$  in (3.16) we find that

$$\lim_{t \rightarrow \infty} \frac{d}{dt} y = -V'(\bar{\phi}) < 0.$$

Hence, there exists  $t'' > t'$  such that  $\frac{d}{dt} y < -V'(\bar{\phi})/2$  for all  $t \geq t''$ . This implies

$$y(t) - y(t'') = \int_{t''}^t \left( \frac{d}{dt} y \right) dt < -\frac{V'(\bar{\phi})}{2} (t - t''),$$

that is,  $y(t)$  takes negative values with arbitrary large modulus as  $t$  increases, which is not possible since  $\lim_{t \rightarrow \infty} y(t) = 0$ .

Hence, if  $\phi > 0$  for all  $t > t'$ , we have that  $\lim_{t \rightarrow \infty} \phi = +\infty$ . Similarly, when  $\phi < 0$  for all  $t > t'$ , we have  $\lim_{t \rightarrow \infty} \phi = -\infty$ . ■

From this we conclude that, if initially  $3H(t_0)^2 < \min \{\lim_{\phi \rightarrow \infty} V(\phi), \lim_{\phi \rightarrow -\infty} V(\phi)\}$ , then,  $\lim_{t \rightarrow \infty} H(t) = 0$ . Indeed, we have that  $\lim_{t \rightarrow \infty} \phi$  is equal to  $+\infty$ ,  $0$  or  $-\infty$ . If  $\lim_{t \rightarrow \infty} \phi = +\infty$ , then from the restriction (3.18), follows

$$3\eta^2 = \lim_{t \rightarrow \infty} V(\phi(t)) = \lim_{\phi \rightarrow \infty} V(\phi) > 3H(t_0)^2.$$

This is impossible since  $H(t)$  is a decreasing function and  $H(t_0) \geq \eta$ . In the same way,  $\lim_{t \rightarrow \infty} \phi = -\infty$  leads to a contradiction. Then,  $\lim_{t \rightarrow \infty} \phi = 0$  and this implies  $\lim_{t \rightarrow \infty} V(\phi(t)) = 0$ , and again by (3.18),  $\lim_{t \rightarrow \infty} H(t) = 0$ .

The interpretation of these results is clear.

If the potential has a local minimum at zero, if the derivative of the potential is bounded in the same set where the potential itself is, and provided the derivative of the logarithm of the coupling function is bounded by above, then, the energy density of DM and the kinetic energy density of DE tends to zero as the time goes forward. In this case the energy density of the Universe will be dominated by the potential energy of DE. Hence, the Universe would be expand forever in a de Sitter phase.

With the above hypotheses and with the additional assumption of  $V(\phi)$  being strictly decreasing (increasing) if  $\phi < 0$  ( $\phi > 0$ ) it is proved (in a similar way as in Proposition 3 in [404]) that the scalar field can be either zero or divergent into the future (the former case holds if the Hubble scalar vanishes asymptotically).

In order to complement the former ideas, we must remark that if the potential is non negative (and with no necessarily a local minimum at  $(0, 0)$ ) having continuous derivative (bounded in the same set where the potential itself is); and assuming that the potential  $V(\phi(t))$ , strictly decreasing as a function of  $t$ , unbounded when  $t \rightarrow \infty$ . Then, the cosmological model enters a de Sitter expansion, characterized by divergences of the scalar field into the future. If additionally the potential as a function of  $\phi$  vanishes asymptotically into the future, the Hubble scalar vanishes too. This fact is true for exponential potentials.

**Proposition 6** *Suppose that there exists a nonzero constant  $K$ , such that  $\chi'(\phi)/\chi(\phi) \leq 2K/(2 - \gamma)(4 - 3\gamma)$ . Let  $V$  be a potential function with the properties:*

1.  $V \geq 0$  and  $\lim_{t \rightarrow \infty} V(\phi) = +\infty$ .
2.  $V'$  is continuous and  $V'(\phi) < 0$ .
3. If  $A \subset \mathbb{R}$  is such that  $V$  is bounded in  $A$ , Then,  $V'(\phi)$  is bounded in  $A$ .

Then,  $\lim_{t \rightarrow \infty} \rho = 0 = \lim_{t \rightarrow \infty} y$ , and  $\lim_{t \rightarrow \infty} \phi = +\infty$ .

**Proof.** From equation (3.15), the set  $\rho > 0$  is invariant under the flow of (3.14)-(3.17) with restriction (3.18); besides  $\rho$  is different from zero if  $\rho(t_0)$  is different from zero at the initial time. From this fact we have that  $H$  is never zero (do not changes of sign) since by (3.18),  $3H(t)^2 \geq \rho(t) > 0$  for all  $t > t_0$ , then,  $H$  is always nonnegative if initially is nonnegative. Besides, by equation (3.14), follows that  $H$  is decreasing, then  $\exists \lim_{t \rightarrow \infty} H(t) = \eta \geq 0$  and

$$\frac{1}{2} \int_{t_0}^{\infty} (y^2 + \gamma\rho) dt = H(t_0) - \eta < +\infty.$$

As in proposition 4, the total time derivative of  $y^2 + \gamma\rho$  is bounded. Hence  $\lim_{t \rightarrow \infty} \rho = 0 = \lim_{t \rightarrow \infty} y$ .

It can be proved that  $\lim_{t \rightarrow \infty} \phi = +\infty$  in the same way as proved in 5.

From equation (3.18) we have that  $\lim_{t \rightarrow \infty} V(\phi) = 3\eta^2$ . Since  $V$  is strictly decreasing with respect to  $\phi$ ; then  $V(\phi) > \lim_{\phi \rightarrow \infty} V(\phi)$  for all  $\phi$ , therefore  $\lim_{t \rightarrow \infty} V(\phi(t)) \geq \lim_{\phi \rightarrow \infty} V(\phi)$ . We will consider two cases:

1. If  $\lim_{t \rightarrow \infty} V(\phi(t)) = \lim_{\phi \rightarrow \infty} V(\phi)$ , by the continuity of  $V$  is obvious that  $\lim_{t \rightarrow \infty} \phi = +\infty$ ;
2. If  $\lim_{t \rightarrow \infty} V(\phi(t)) > \lim_{\phi \rightarrow \infty} V(\phi)$ , then, there exists a unique  $\bar{\phi}$  such that

$$\lim_{t \rightarrow \infty} V(\phi(t)) = V(\bar{\phi}).$$

Since  $V$  is continuous and strictly decreasing follows that

$$\lim_{t \rightarrow \infty} \phi = \bar{\phi}.$$

From equation (3.16) follows that

$$\lim_{t \rightarrow \infty} \frac{d}{dt} y = -V'(\bar{\phi}) > 0,$$

therefore, exists  $t'$  such that  $\frac{d}{dt} y > -V'(\bar{\phi})/2$  for all  $t \geq t'$ . Form this fact we conclude that

$$y(t) - y(t') > -\frac{V'(\bar{\phi})}{2}(t - t'),$$

which is impossible since  $\lim_{t \rightarrow \infty} y(t) = 0$ . Finally  $\lim_{t \rightarrow \infty} \phi = +\infty$ .

■

If additionally, the potential is such that  $\lim_{\phi \rightarrow \infty} V(\phi) = 0$ , then we conclude that  $H \rightarrow 0$  as  $t \rightarrow \infty$ .

Now, let us consider the general case by including radiation. Our purpose is to formulate a proposition that extent in some way (we are considering only flat FRW models) the proposition 1 of [407], which gives a characterization of the future attractor of the system (3.2), (3.3), (3.4), (3.5).

Let us formalize notion of degenerate local minimum introduced in [407]:

**Definition 24** *The function  $V(\phi)$  is said to have a degenerate local minimum at  $\phi_*$  if*

$$V'(\phi), V''(\phi), \dots V^{(2n-1)}(\phi)$$

*vanish at  $\phi_*$ , and  $V^{(2n)}(\phi_*) > 0$ , for some integer  $n$ .*

**Proposition 7** *Suppose that  $V(\phi) \in C^2(\mathbb{R})$  satisfies the following conditions <sup>2</sup>*

- (i) *The (possibly empty) set  $\{\phi : V(\phi) < 0\}$  is bounded;*
- (ii) *The (possibly empty) set of singular points of  $V(\phi)$  is finite.*

*Let  $\phi_*$  a strict local minimum for  $V(\phi)$ , possibly degenerate, with non-negative critical value. Then  $\mathbf{p}_* := \left( \phi_*, y_* = 0, \rho_* = 0, \rho_r = 0, H = \sqrt{\frac{V(\phi_*)}{3}} \right)$  is an asymptotically stable singular point for the flow of (3.7)-(3.11).*

**Proof.**

We adapt the demonstration in [407] (for flat FRW cosmologies) to the case where radiation is considered. <sup>3</sup>

First let us consider the case  $V(\phi_*) > 0$ . Let  $\tilde{V} > V(\phi_*)$  be a regular value for  $V$  such that the connected component of  $V^{-1}((-\infty, \tilde{V}])$  that contains  $\phi_*$  is a compact set in  $\mathbb{R}$ . Let us denote this set by  $A$  and define  $\Psi$  as

$$\Psi = \left\{ (\phi, y, \rho, \rho_r, H) : \phi \in A, \epsilon + \rho \leq \tilde{V}, \rho \geq 0, \rho_r \in [0, \tilde{W}] \right\},$$

where  $\tilde{W}$  is a positive constant. We can show that  $\Psi$  is a compact set as follows.

- (i)  $\Psi$  is a closed set in  $\mathbb{R}^5$ ;
- (ii)  $V(\phi_*) \leq V(\phi) \leq \tilde{V}, \forall \phi \in A$ ;
- (iii)  $\frac{1}{2}y^2 + V(\phi_*) \leq \frac{1}{2}y^2 + V(\phi) + \rho \leq \tilde{V}$ , and therefore  $y$  is bounded;
- (iv)  $\rho \leq \tilde{V} - \frac{1}{2}y^2 - V(\phi) \leq \tilde{V} - V(\phi_*)$  and then  $\rho$  is bounded;

---

<sup>2</sup>See assumptions 1 in [407].

<sup>3</sup>From physical considerations we can neglect radiation for the analysis of the future attractor, but we prefer to offer the complete **Proof**.

(v) Finally, from (3.12),  $\frac{V(\phi_*)}{3} \leq H^2 \leq \tilde{V} + \tilde{W}$ .

Let  $\Psi_+ \subseteq \Psi$  be the connected component of  $\Psi$  containing  $\mathbf{p}_*$ . Following similar arguments as in [407] it can be proved that  $\Psi_+$  is positively invariant with respect to (3.7)-(3.11), i.e, all the solutions with initial data at  $\Psi_+$  remains at  $\Psi_+$  for all  $t > 0$ . Indeed, let  $\mathbf{x}(t)$  be such a solution and

$$\bar{t} = \sup \{t > 0 : H(t) > 0\} \in \mathbb{R} \cup \{+\infty\}.$$

When  $t < \bar{t}$ , equation (3.13) imply that  $\epsilon + \rho$  decreases monotonically (cf remark 1). Moreover, it can be proved by contradiction that

$$\phi(t) \in A \forall t < \bar{t}, \quad (3.20)$$

otherwise there would exists some  $t < \bar{t}$  such that  $V(\phi(t)) > \tilde{V}$ , but then

$$\tilde{V} < V(\phi(t)) \leq \frac{1}{2}y(t)^2 + V(\phi(t)) + \rho(t) \leq \tilde{V},$$

a contradiction. Thus, (3.20) holds. But since  $\rho_r \geq 0$  along the flow (cf remark 1), it follows that

$$H(t)^2 \geq \frac{1}{3} \left( \frac{1}{2}y(t)^2 + V(\phi(t)) + \rho(t) \right) \geq \frac{V(\phi(t))}{3} \geq \frac{V(\phi_*)}{3} > 0.$$

We have proved that as long as  $H$  remains positive, it is strictly bounded away from zero; thus  $\bar{t} = +\infty$ , and from this can be deduced that  $\mathbf{x}(t)$  remains in  $\Psi_+$  for all  $t > 0$ .

From all the above  $\Psi_+$  satisfies the hypothesis of LaSalle's invariance theorem (see [361]; theorem 8.3.1 in [358], p. 111). If we consider the monotonic decreasing functions  $\epsilon + \rho$  and  $\rho_r$  defined in  $\Psi_+$  then follows that, every solution with initial data at  $\Psi_+$  must be such that  $H(y^2 + \gamma\rho) \rightarrow 0$  and  $H\rho_r \rightarrow 0$  as  $t \rightarrow +\infty$ . Since  $H$  is strictly bounded away from zero in  $\Psi_+$  follows that  $(y, \rho, \rho_r) \rightarrow (0, 0, 0)$  and  $H^2 - \frac{V(\phi)}{3} \rightarrow 0$  as  $t \rightarrow +\infty$ .

Since  $H$  is monotone decreasing (cf remark 1) and it is bounded away from zero it must have a limit. This means that  $V(\phi)$  also admits a limit. This limit has to be  $V(\phi_*)$ ; otherwise  $V'(\phi)$  would tend to a positive value and so would the righthand side of (3.8), a contradiction. Therefore the solution approaches the singular point  $\mathbf{p}_*$ .

If  $V(\phi_*) = 0$ , the above argument can be easily adapted. In this case the set  $\Omega$  is connected and we choose  $\Omega_+$  to be its subset characterized by the property  $H \geq 0$ . The only point in  $\Omega_+$  with  $H = 0$  is exactly the singular point  $\mathbf{p}_*$ , and so if  $H(t) \rightarrow 0$  the solution is forced to approach the equilibrium since  $H$  is monotone; if by contradiction  $H(t)$  had a strictly positive limit, we could argue as before to find  $y \rightarrow 0$ ,  $\rho \rightarrow 0$  and  $\rho_r \rightarrow 0$  and so  $H$  must necessarily converge to zero. ■

### 3.4 Early time behavior in the invariant set $\rho_r = 0$

In order to analyze the initial singularity (and also, the late time behavior) it is convenient to normalize the variables, since in the vicinity of an hypothetical initial singularity, the physical variables would typically diverge, whereas at late times they commonly vanish [411]. In this section we rewrite equations (3.2), (3.3), (3.4), (3.5) as an autonomous system defined on a state space by introducing Hubble-normalized variables. These variables satisfy an inequality arising from the Friedmann equation (3.6). We analyze the cosmological model by investigating the flow of the autonomous system in a phase space by using dynamical systems tools.

Let us introduce the following normalized variables

$$x_1 = \frac{\dot{\phi}}{\sqrt{6}H}, \quad x_2 = \frac{\sqrt{\rho}}{\sqrt{3}H}, \quad x_3 = \frac{1}{H} \quad (3.21)$$

and the time coordinate

$$d\tau = 3H dt. \quad (3.22)$$

Using the field equations (3.14)-(3.17) we get that the variables (3.21) and the scalar field  $\phi$  evolve with respect  $\tau$  as

$$\phi' = \sqrt{\frac{2}{3}}x_1, \quad (3.23)$$

$$x_1' = x_1^3 + \frac{1}{2}(x_2^2\gamma - 2)x_1 - \frac{x_3^2}{3\sqrt{6}}\frac{dV(\phi)}{d\phi} + \frac{(4-3\gamma)x_2^2}{2\sqrt{6}}\frac{d\ln\chi(\phi)}{d\phi}, \quad (3.24)$$

$$x_2' = \frac{1}{2}x_2(2x_1^2 + (x_2^2 - 1)\gamma) - \frac{(4-3\gamma)x_1x_2}{2\sqrt{6}}\frac{d\ln\chi(\phi)}{d\phi}, \quad (3.25)$$

$$x_3' = \frac{1}{2}x_3(2x_1^2 + x_2^2\gamma), \quad (3.26)$$

where the prime denotes derivative with respect  $\tau$ . This is an autonomous system where the variables are subject to the constraint

$$x_1^2 + x_2^2 + \frac{1}{3}x_3^2V(\phi) = 1. \quad (3.27)$$

From the hypotheses  $V \in C^3, V(\phi) > 0, \chi \in C^3, \chi(\phi) > 0, 0 < \gamma < 2, \gamma \neq \frac{4}{3}$  follows that (3.23)-(3.26) defines a dynamical system of class  $C^2$  in  $\mathbb{R}^4$ .

**Proposition 8** *The sets defined by*

$$\Sigma_T := \left\{ \mathbf{p} = (\phi, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + \frac{1}{3}x_3^2V(\phi) = 1 \right\}, \quad (3.28)$$



$$V_{jk} := \{\mathbf{p} \in \mathbb{R}^4 | (-1)^k x_j > 0\} \cap \Sigma_T, \quad (3.29)$$

and

$$U_j := \{\mathbf{p} \in \mathbb{R}^4 | x_j = 0\} \cap \Sigma_T, \quad (3.30)$$

with  $j = 2, 3; k = 1, 2$ ; are invariant sets for the flow of (3.23)-(3.26) defined in  $\mathbb{R}^4$ .

**Proof.** To prove these results we use the proposition 2.

1. To prove the invariance of  $\Sigma_T$  we define

$$Z : \mathbb{R}^4 \rightarrow \mathbb{R}, \mathbf{p} \rightarrow x_1^2 + x_2^2 + \frac{1}{3}x_3^2 V(\phi) - 1 \quad (3.31)$$

and

$$\alpha : \mathbb{R}^4 \rightarrow \mathbb{R}, \mathbf{p} \rightarrow 2x_1^2 + x_2^2 \gamma. \quad (3.32)$$

Observe that  $Z$  and  $\alpha$  satisfy the hypothesis of proposition 2. Thus  $\Sigma_T = \{\mathbf{p} \in \mathbb{R}^4 | Z(\mathbf{p}) = 0\}$  is invariant.

2. To prove the invariance of  $V_{2k}$ ,  $k = 1, 2$  and  $U_2$  we define

$$Z : \Sigma_T \rightarrow \mathbb{R}, \mathbf{p} \rightarrow x_2 \quad (3.33)$$

and

$$\alpha : \Sigma_T \rightarrow \mathbb{R}, \mathbf{p} \rightarrow \frac{1}{2} (2x_1^2 + (x_2^2 - 1) \gamma) - \frac{(4 - 3\gamma)x_1}{2\sqrt{6}} \frac{d \ln \chi(\phi)}{d\phi}. \quad (3.34)$$

Observe that  $Z$  and  $\alpha$  satisfy the hypothesis of proposition 2. Thus  $V_{21} = \{\mathbf{p} \in \Sigma_T | Z(\mathbf{p}) < 0\}$ ,  $V_{22} = \{\mathbf{p} \in \Sigma_T | Z(\mathbf{p}) > 0\}$  and  $U_2 = \{\mathbf{p} \in \Sigma_T | Z(\mathbf{p}) = 0\}$  are invariant.

3. To prove the invariance of  $V_{3k}$ ,  $k = 1, 2$  and  $U_3$  we define

$$Z : \Sigma_T \rightarrow \mathbb{R}, \mathbf{p} \rightarrow x_3 \quad (3.35)$$

and

$$\alpha : \Sigma_T \rightarrow \mathbb{R}, \mathbf{p} \rightarrow \frac{1}{2} (2x_1^2 + x_2^2 \gamma). \quad (3.36)$$

Observe that  $Z$  and  $\alpha$  satisfy the hypothesis of proposition 2. Thus  $V_{31} = \{\mathbf{p} \in \Sigma_T | Z(\mathbf{p}) < 0\}$ ,  $V_{32} = \{\mathbf{p} \in \Sigma_T | Z(\mathbf{p}) > 0\}$  and  $U_3 = \{\mathbf{p} \in \Sigma_T | Z(\mathbf{p}) = 0\}$  are invariant.

■

According to proposition 8, the invariant set  $\Sigma_T$  acts as an independent dynamical object and thus will be suffice to consider the autonomous system (3.23)-(3.26) defined in  $\Sigma_T$ . Observe that the sets  $V_{jk}, U_j, j = 2, 3; k = 1, 2$ ; are invariant sets for the flow of (3.23)-(3.26) defined in  $\Sigma_T$ .

In the following we consider the **notations**:  $\|\cdot\|$  denotes the Euclidean vector norm;  $\mathbb{D}^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < 1\}$  denotes the n-dimensional unitary disc.

### 3.4.1 The Topological Properties of the Phase Space

In the following we discuss the topological properties of  $\Sigma_T$  resulting the

**Proposition 9**  $\Sigma_T$  is a topological manifold (without boundary).

**Proof.** First, let us prove that  $\Sigma_T$  has dimension less than 4. For this purpose it is sufficient to prove that  $\Sigma_T$  is not an open set with respect to the usual topology of  $\mathbb{R}^4$ . The mapping (3.31) is continuous for all  $\mathbf{p} \in \mathbb{R}^4$ . Let  $\mathbf{p}_0$  an arbitrary point of  $\mathbb{R}^4 \setminus \Sigma_T$ . Then,  $Z(\mathbf{p}_0) = c \neq 0$  where  $c$  is a constant. Since  $Z$  is continuous and it is defined for all  $\mathbf{p}$ , follows that there exists a real number  $\delta > 0$  such that for all  $\mathbf{p}$  of  $S_\delta(\mathbf{p}_0) = \{\mathbf{p} \in \mathbb{R}^4 : \|\mathbf{p} - \mathbf{p}_0\| < \delta\}$ , we have  $\|Z(\mathbf{p}) - Z(\mathbf{p}_0)\| < \frac{1}{2}|c|$ . Then,  $\|Z(\mathbf{p}) - c\| < \frac{1}{2}|c|$ . From this follows that  $Z(\mathbf{p}) \neq 0$  for all  $\mathbf{p} \in S_\delta(\mathbf{p}_0)$ , where  $S_\delta(\mathbf{p}_0)$  is an open set of  $\mathbb{R}^4$  contained in  $\mathbb{R}^4 \setminus \Sigma_T$ . We conclude that  $\mathbb{R}^4 \setminus \Sigma_T$  is an open set of  $\mathbb{R}^4$ ; thus,  $\Sigma_T$  is a closed subset of  $\mathbb{R}^4$ . Second, since  $\Sigma_T$  is a closed set with respect the usual topology of  $\mathbb{R}^4$ ; it is a Hausdorff space equipped with a numerable basis. The rest of the proof requires the construction of a set of local charts.

The sets  $V_{jk}, j = 1, 2, 3$  defined by the formula (3.29) are open sets of  $\Sigma_T$  with respect to the induced topology in  $\Sigma_T$  by the usual topology of  $\mathbb{R}^4$ , which cover  $\Sigma_T$ .

Let us define the maps:

$$\begin{aligned} h_{1k} : V_{1k} &\rightarrow \mathbb{R}^3, \\ \mathbf{p} \rightarrow h_{1k}(\mathbf{p}) &= \left( \phi, x_2, \sqrt{\frac{V(\phi)}{3}}x_3 \right) = (\xi_1, \xi_2, \xi_3), \quad k = 1, 2, \end{aligned} \quad (3.37)$$

which satisfy  $h_{1k}(V_{1k}) = \mathbb{R} \times \mathbb{D}^2$  with inverse given by

$$\begin{aligned} h_{1k}^{-1} : \mathbb{R} \times \mathbb{D}^2 &\rightarrow V_{1k}, \\ \xi \rightarrow h_{1k}^{-1}(\xi) &= \left( \xi_1, (-1)^k \sqrt{1 - \xi_2^2 - \xi_3^2}, \xi_2, \sqrt{\frac{3}{V(\xi_1)}}\xi_3 \right), \quad k = 1, 2; \end{aligned} \quad (3.38)$$

$$h_{2k} : V_{2k} \rightarrow \mathbb{R}^3,$$

$$\mathbf{p} \rightarrow h_{1k}(\mathbf{p}) = \left( \phi, x_1, \sqrt{\frac{V(\phi)}{3}} x_3 \right) = (\xi_1, \xi_2, \xi_3), \quad k = 1, 2; \quad (3.39)$$

which satisfy  $h_{1k}(V_{1k}) = \mathbb{R} \times \mathbb{D}^2$ , with inverse given by

$$h_{2k}^{-1} : \mathbb{R} \times \mathbb{D}^2 \rightarrow V_{2k},$$

$$\xi \rightarrow h_{2k}^{-1}(\xi) = \left( \xi_1, \xi_2, (-1)^k \sqrt{1 - \xi_2^2 - \xi_3^2}, \sqrt{\frac{3}{V(\xi_1)}} \xi_3 \right), \quad k = 1, 2; \quad (3.40)$$

and

$$h_{3k} : V_{3k} \rightarrow \mathbb{R}^3,$$

$$\mathbf{p} \rightarrow h_{3k}(\mathbf{p}) = (\phi, x_1, x_2) = (\xi_1, \xi_2, \xi_3), \quad k = 1, 2 \quad (3.41)$$

which satisfy  $h_{1k}(V_{1k}) = \mathbb{R} \times \mathbb{D}^2$ , with inverse given by

$$h_{3k}^{-1} : \mathbb{R} \times \mathbb{D}^2 \rightarrow V_{3k},$$

$$\xi \rightarrow h_{3k}^{-1}(\xi) = \left( \xi_1, \xi_2, \xi_3, (-1)^k \sqrt{\frac{3(1 - \xi_2^2 - \xi_3^2)}{V(\xi_1)}} \right), \quad k = 1, 2. \quad (3.42)$$

Since  $V(\phi)$  is positive and differentiable of class  $C^3$ , follows that the above functions and they inverses are of class  $C^3$  (and then continuous). Hence, all the above applications are homeomorphism (in fact they are diffeomorphism of class  $C^3$ ).

From the previous characterization of  $\Sigma_T$  follows that it is a topological manifold (without boundary) of dimension 4 immersed in  $\mathbb{R}^4$ , since  $id : \Sigma_T \rightarrow \mathbb{R}^4, \mathbf{p} \rightarrow id(\mathbf{p}) = \mathbf{p}$  is a diffeomorphic immersion. ■

**Observation.** The system (3.23)-(3.26) is invariant under the coordinate transformations  $x_2 \rightarrow -x_2$  and  $x_3 \rightarrow -x_3$ . Thus the analysis can be simplified considerably if we restrict the flow to the invariant set

$$\Sigma := \left\{ (\phi, x_1, x_2, x_3) \in \mathbb{R}^2 \times (\mathbb{R}_+^0)^2 \right\} \cap \Sigma_T = V_{22} \cup V_{32} \cup U_2 \cup U_3. \quad (3.43)$$

where we have used the notation  $\mathbb{R}_+^0 := \{x \in \mathbb{R} | x \geq 0\}$ , and  $\mathbb{R}_+ := \{x \in \mathbb{R} | x > 0\}$ .

**Proposition 10**  $\Sigma$  is topological manifold with boundary

**Proof.** Using the same arguments as in the proof of proposition 9, we can prove that  $\Sigma$  is not an open set with respect to the induced topology as subset of  $\mathbb{R}^2 \times \mathbb{R}_+^2$ ; thus its dimension should be less than 4. Let us construct a set of local charts as follows.

Let be defined the sets  $W_j = \{\mathbf{p} \in \Sigma, x_j > 0\}$ ,  $j = 2, 3$ . These sets are open with respect to the topology induced in  $\Sigma$  by the usual topology of  $\mathbb{R}^2 \times \mathbb{R}_+^2$ .

Let be defined the maps

$$h_2 : W_2 \rightarrow \mathbb{H}^3, \mathbf{p} \rightarrow h_2(\mathbf{p}) = \left( \phi, x_1, \sqrt{\frac{V(\phi)}{3}} x_3 \right) = (\xi_1, \xi_2, \xi_3) \quad (3.44)$$

and

$$h_3 : W_3 \rightarrow \mathbb{H}^3, \mathbf{p} \rightarrow h_3(\mathbf{p}) = (\phi, x_1, x_2) = (\xi_1, \xi_2, \xi_3) \quad (3.45)$$

Observe that  $h_j(\mathbf{p}) = \mathbb{R} \times (\mathbb{H}^2 \cap \mathbb{D}^2)$  which are open sets of  $\mathbb{H}^3$  with respect to the induced topology. The inverse functions of  $h_1$  and  $h_2$  are given by

$$\begin{aligned} h_2^{-1} : \mathbb{R} \times (\mathbb{H}^2 \cap \mathbb{D}^2) &\rightarrow W_2, \\ \xi &\rightarrow h_2^{-1}(\xi) = \left( \xi_1, \xi_2 \sqrt{1 - \xi_2^2 - \xi_3^2}, \sqrt{\frac{3}{V(\xi_1)}} \xi_3 \right) \end{aligned} \quad (3.46)$$

and

$$\begin{aligned} h_3^{-1} : \mathbb{R} \times (\mathbb{H}^2 \cap \mathbb{D}^2) &\rightarrow W_3, \\ \xi &\rightarrow h_3^{-1}(\xi) = \left( \xi_1, \xi_2, \xi_3, \sqrt{\frac{3(1 - \xi_2^2 - \xi_3^2)}{V(\xi_1)}} \right) \end{aligned} \quad (3.47)$$

From the hypotheses on  $V$  follows that all these functions are differentiable of class  $C^3$ . Hence,  $(W_2, h_2), (W_3, h_3)$  are local charts. Since the local charts  $(W_2, h_2), (W_3, h_3)$  do not cover the sets with  $x_2 = x_3 = 0$  we define the sets  $W_1^\pm = \{\mathbf{p} \in \Sigma : x_1 = \pm 1, x_2 = x_3 = 0\}$ ,  $W_1 = W_1^+ \cup W_1^-$ . From the equalities  $W_1^+ = V_{12} \cap U_2 \cap U_3$  and  $W_1^- = V_{11} \cap U_2 \cap U_3$  where  $V_{1k} = \{\mathbf{p} \in \mathbb{R}^4 | (-1)^k x_1 > 0\} \cap \Sigma$  follow that  $W_1^\pm$  are open sets with respect to the topology induced in  $\Sigma$  by the usual topology of  $\mathbb{R}^2 \times \mathbb{R}_+^2$  which are disjoint copies of  $\mathbb{R}$ . ■

We can prove that  $W_2$  and  $W_3$  are topological manifolds with boundary. In order to do so we observe that, since  $V$  is positive, the points  $\mathbf{p} \in W_2$  transforms by  $h_2$  is points with  $\xi_3 := \sqrt{\frac{V(\phi)}{3}} x_3 = 0$ , if, and only if,  $x_3 = 0$ . In an analogous way, it is easy to see that the points  $\mathbf{p} \in W_3$  transforms by  $h_3$  is points with  $\xi_3 := x_2 = 0$ , if, and only if,  $x_2 = 0$ . Thus, the boundaries of  $W_2$  and  $W_3$  are given respectively by

$$\begin{aligned} \partial W_2 &= \{\mathbf{p} = (\phi, x_1, x_2, x_3) \in W_2 : h_2(\mathbf{p}) \in \mathbb{R}^2 \times \{0\}\} \\ &= \{\mathbf{p} \in W_2 : x_3 = 0\} \end{aligned} \quad (3.48)$$

$$\begin{aligned} \partial W_3 &= \{\mathbf{p} = (\phi, x_1, x_2, x_3) \in W_3 : h_3(\mathbf{p}) \in \mathbb{R}^2 \times \{0\}\} \\ &= \{\mathbf{p} \in W_3 : x_2 = 0\}. \end{aligned} \quad (3.49)$$

Let us define the sets

$$\begin{aligned} (\partial\Sigma)_1 &= \partial W_2 \cup W_1 = \{\mathbf{p} \in \Sigma : x_3 = 0\} \\ &= \{(\phi, x_1, x_2) \in \mathbb{H}^3 : x_1^2 + x_2^2 = 1\} \end{aligned} \quad (3.50)$$

and

$$\begin{aligned} (\partial\Sigma)_2 &= \partial W_3 \cup W_1 = \{\mathbf{p} \in \Sigma : x_2 = 0\} \\ &= \left\{ (\phi, x_1, x_3) \in \mathbb{H}^3 : x_1^2 + \frac{V(\phi)}{3} x_3^2 = 1 \right\}. \end{aligned} \quad (3.51)$$

Defining the maps

$$g : (\partial\Sigma)_1 \rightarrow \mathbb{R} \times [-1, 1], (\phi, x_1, x_2) \rightarrow (\phi, x_1) = (\xi_1, \xi_2), \quad (3.52)$$

$$g^{-1} : \mathbb{R} \times [-1, 1] \rightarrow (\partial\Sigma)_1, (\xi_1, \xi_2) \rightarrow \left( \xi_1, \xi_2, \sqrt{1 - \xi_2^2} \right); \quad (3.53)$$

$$h : (\partial\Sigma)_2 \rightarrow \mathbb{R} \times [-1, 1], (\phi, x_1, x_3) \rightarrow (\phi, x_1) = (\xi_1, \xi_2), \quad (3.54)$$

$$h^{-1} : \mathbb{R} \times [-1, 1] \rightarrow (\partial\Sigma)_2, (\xi_1, \xi_2) \rightarrow \left( \xi_1, \xi_2, \sqrt{\frac{3(1-\xi_2^2)}{V(\xi_1)}} \right); \quad (3.55)$$

and

we can show that  $(\partial\Sigma)_1$  and  $(\partial\Sigma)_2$  are 2-dimensional manifolds with boundaries given by  $W_1$ .

It is easy to show the identities

$$\Sigma = \text{Int}(\Sigma \setminus W_1) \cup (\partial\Sigma)_1 \cup (\partial\Sigma)_2 \quad (3.56)$$

$$\text{Int}\Sigma = \text{Int}(\Sigma \setminus W_1) = \{\mathbf{p} \in \Sigma : x_2 > 0, x_3 > 0\} \quad (3.57)$$

From the above arguments and expression (3.56) we have the following.

**Remark 2** • *The interior of  $\Sigma$  is given by (3.57). It is a 3- dimensional manifold without boundary.*

- *The boundary of  $\Sigma$ ,  $\partial\Sigma$ , is the union of two 2-dimensional topological manifolds with boundary given by  $(\partial\Sigma)_1 = \{\mathbf{p} \in \Sigma : x_3 = 0\}$  and  $(\partial\Sigma)_2 = \{\mathbf{p} \in \Sigma : x_2 = 0\}$ , contained in  $\mathbb{R} \times \mathbb{R}_+^0$ .*
- *$(\partial\Sigma)_1$  and  $(\partial\Sigma)_2$  share the same boundary (which is a union of two disjoint a copies of  $\mathbb{R}$ ) given by  $W_1 = \{\mathbf{p} \in \Sigma | x_1 = 1\} \cup \{\mathbf{p} \in \Sigma | x_1 = -1\}$ .*

**Lemma 1** *Suppose that  $0 < \chi(\phi) < +\infty$  for any compact set. Then, for all  $p \in \text{Int}\Sigma$  the  $\alpha$ - and  $\omega$ -limit sets of  $p$  are such that  $\alpha(p) \subset (\partial\Sigma)_1$  and  $\omega(p) \subset (\partial\Sigma)_2$ .*

**Proof.** By the proposition 2 we have that  $Int\Sigma = \{(\phi, x_1, x_2, x_3) \in \Sigma : x_2 > 0, x_3 > 0\}$  is an invariant set of the flow of (3.23)-(3.26). Let be defined on  $Int\Sigma$  the function

$$Z(\phi, x_1, x_2, x_3) = \left(\frac{x_2}{x_3}\right)^2 \chi(\phi)^{2-\frac{3\gamma}{2}}.$$

The function  $Z$  is a monotone decreasing function (in the direction of the flow) in  $Int\Sigma$ , since its directional derivative through the flow is  $Z' = -\gamma Z$ . The rank of  $Z$  is  $(0, \infty)$ . Let be  $s \in \partial\Sigma$ . From the hypothesis about  $\chi$  follows that it cannot not be zero or infinite unless  $|\phi| \rightarrow \infty$ . Then, it is verified that  $Z(s) \rightarrow 0$  as  $s \rightarrow (\partial\Sigma)_2$  and  $Z(s) \rightarrow \infty$  as  $s \rightarrow (\partial\Sigma)_1$ . Hence, applying the Monotonicity Principle (theorem 18) follows the required result.

### 3.4.2 The flow on the invariant set $(\partial\Sigma)_1$

The flow on  $(\partial\Sigma)_1$  is governed by the differential equations:

$$\phi' = \sqrt{\frac{2}{3}}x_1, \quad (3.58)$$

$$x_1' = \frac{1}{2}(1 - x_1^2) \left( x_1(\gamma - 2) + \frac{(4 - 3\gamma)}{\sqrt{6}} \frac{\chi'(\phi)}{\chi(\phi)} \right), \quad (3.59)$$

If  $\mathbf{p} \in (\partial\Sigma)_1$  then  $x_1^2 + x_2^2 = 1$ , which implies  $|x_1| \leq 1$ . The equality holds if and only if  $x_2 = 0$ . It is easy to prove that the set

$$W_1 = \{\mathbf{p} = (\phi, x_1, x_2, x_3) \in \Sigma : |x_1| = 1\}$$

is an invariant set of the flow of (3.23)-(3.26) in  $\Sigma$ . Observe that  $\phi$  is unbounded on  $W_1$  since from  $|x_1| = 1$  and by the equation (3.23),

$$\phi' = \pm \sqrt{\frac{2}{3}} \implies \phi = \phi_0 \pm \sqrt{\frac{2}{3}}\tau$$

(which is an unbounded function of  $\tau$ ).

In the set  $(\partial\Sigma)_1$  there exist a (possibly empty) family of singular points with  $\phi$  bounded  $Q := \{\mathbf{q} := (\phi, x_1) = (q_1, 0) \in \bar{S} : \chi'(q_1) = 0\}$ .<sup>4</sup> The eigenvalues of the matrix of derivatives evaluated at  $\mathbf{q} \in Q$  are

$$\Delta_1 \pm \sqrt{\Delta_1^2 + \Delta_2 \frac{\chi''(q_1)}{\chi(q_1)}} \quad (3.60)$$

---

<sup>4</sup>For definiteness we are assuming that  $\chi(\phi_1) \neq 0$ . Also, we are assuming that  $\chi$  admits only isolated singular points.

with  $\Delta_1 = (-2 + \gamma)/4 < 0$ , and  $\Delta_2 = (4 - 3\gamma)/6$ . The local dynamical character of the singular point  $\mathbf{q} \in Q$  on the invariant set  $(\partial\Sigma)_1$  is as follows <sup>5</sup>:

1.  $\mathbf{q}$  is an stable focus if  $0 < \gamma < 4/3$  and  $\chi''(q_1) < -\frac{\Delta_1^2 \chi(q_1)}{\Delta_2}$  or  $4/3 < \gamma < 2$  and  $\chi''(q_1) > -\frac{\Delta_1^2 \chi(q_1)}{\Delta_2}$ .
2.  $\mathbf{q}$  is an stable node if  $0 < \gamma < 4/3$  and  $-\frac{\Delta_1^2 \chi(q_1)}{\Delta_2} \leq \chi''(q_1) < 0$  or  $4/3 < \gamma < 2$  and  $0 < \chi''(q_1) \leq -\frac{\Delta_1^2 \chi(q_1)}{\Delta_2}$ .
3.  $\mathbf{q}$  is a saddle point if  $0 < \gamma < 4/3$  and  $\chi''(q_1) > 0$  or  $4/3 < \gamma < 2$  and  $\chi''(q_1) < 0$ .
4.  $\mathbf{q}$  is nonhyperbolic, if  $\chi''(q_1) = 0$ , in which case, there exists a 1-dimensional stable manifold which is tangent to the axis  $x_1$  at  $\mathbf{q}$ . There exist also a 1-dimensional center manifold tangent to the line  $(1 - \gamma/2)x_1 - \sqrt{2/3}\phi = 0$  at  $\mathbf{q}$ .

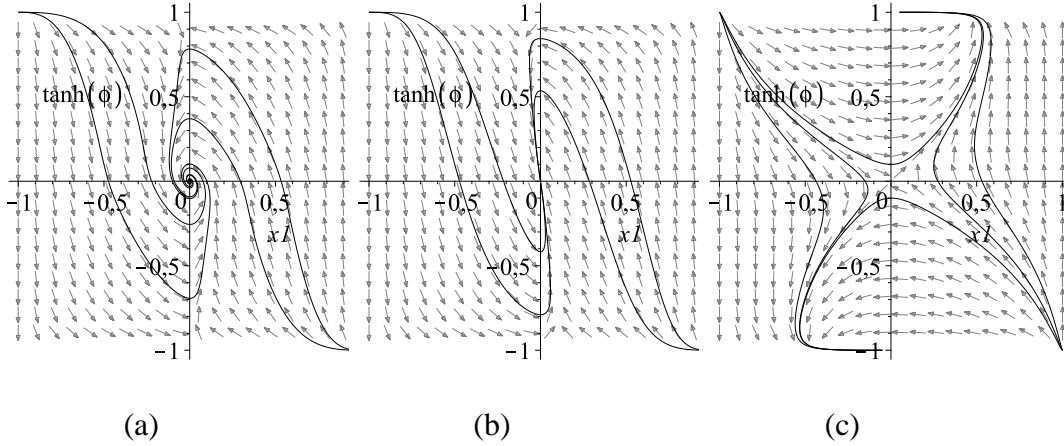


Figure 3.1: Projection of orbits on the phase plane  $(x_1, \varphi)$  for the coupling function (3.61): (a) for  $n = 2$ ,  $\gamma = 1.35$  and  $\chi_0 = 0.05$  the origin is an stable focus. (b) for  $n = 2$ ,  $\gamma = 1.4$  and  $\chi_0 = 3$  the origin is an stable node. (c) for  $n = 2$ ,  $\gamma = 1$  and  $\chi_0 = 0.3$  the origin is saddle point. Notice that the singular points  $(x_1, \varphi) = (0, \pm 1)$  seems to be saddle points for the cases (a) and (b) whereas in the case (c) they are local sinks. Observe that the singular points  $(x_1, \varphi) = (1, -1)$  and  $(y, \varphi) = (-1, 1)$  are in all the cases local sources (which is a suggestive argument in favor for the unboundedness of the scalar field into the past).

### 3.4.2.1 Powerlaw-coupling function

Let us consider the coupling function

$$\chi(\phi) = \frac{\lambda}{n} (\phi^n + \chi_0), \quad \chi_0 > 0, \lambda > 0, n > 1, \quad (3.61)$$

<sup>5</sup>Remember we are assuming that the barotropic index  $\gamma$  satisfies  $0 < \gamma < 2$ .

where  $n$  is an integer number.

In this case the equations (3.58)-(3.59) reduces to

$$\phi' = \sqrt{\frac{2}{3}}x_1, \quad (3.62)$$

$$x_1' = \frac{1}{2}(1 - x_1^2) \left( \frac{n(4 - 3\gamma)\phi^{n-1}}{\sqrt{6}(\phi^n + \chi_0)} + x_1(\gamma - 2) \right). \quad (3.63)$$

Observe that

$$\chi(0) = \frac{\lambda\chi_0}{n} > 0,$$

and

$$\frac{d^m}{d\phi^m}\chi(\phi) = \begin{cases} \lambda\phi^{n-m} \prod_{j=1}^{m-1} (n-j) & , \quad m < n \\ (n-1)! \lambda & , \quad m = n \\ 0 & , \quad m > n \end{cases} \quad (3.64)$$

which implies,  $\frac{d^m\chi}{d\phi^m}(0) = 0$  for all  $m \neq n$  and  $\frac{d^n\chi}{d\phi^n}(0) = (n-1)! \lambda$ .

Hence, if  $n = 2$  then  $\chi(0) = \lambda\chi_0/2$ ,  $\chi'(0) = 0$ ,  $\chi''(0) = \lambda > 0$  (i.e., the coupling function has a local minimum  $\chi(0) = \lambda\chi_0/2 > 0$ ), and by the former analysis, the origin can be either an stable focus if  $4/3 < \gamma < 2$  and  $\chi_0 < -2\Delta_2/\Delta_1^2$ , or an stable node if  $4/3 < \gamma < 2$  and  $\chi_0 \geq -2\Delta_2/\Delta_1^2$ , or a saddle point if  $0 < \gamma < 4/3$ . Observe that if the background is a pressureless dark matter fluid (dust) the origin is a saddle point. If  $n > 2$  then  $\chi'(0) = \chi''(0) = \dots = \chi^{(m)}(0) = 0$ , for  $m < n$ , and  $\mathbf{q} = (0, 0)$  is nonhyperbolic. Thus, there exists a 1-dimensional stable manifold which is tangent to the axis  $x_1$  at the origin. There exist also a 1-dimensional center manifold tangent to the line  $(1 - \gamma/2)x_1 - \sqrt{2/3}\phi = 0$  at the origin.

We have introduced a new scalar field  $\varphi = \tanh(\phi)$  (which takes infinity to a finite value) to make the numerics. Notice the existence of the singular points  $(x_1, \varphi) = (1, -1)$  and  $(x_1, \varphi) = (-1, 1)$  which are in all the cases local sources (which is a suggestive argument in favor for the unboundedness of the scalar field into the past; see figures 3.1, (a)-(c)). Also, if the origin is a saddle point, then the singular points  $(x_1, \varphi) = (0, \pm 1)$  seems to be local sinks (3.1-(c)). However, if the origin is either an stable focus (figure 3.1-(a)) or an stable node (figure 3.1-(b)), the singular points  $(x_1, \varphi) = (0, \pm 1)$  are saddle points. In such a case the orbits spent an infinite amount of time close to the matter dominated universe ( $x_1 = 0 \implies x_2 = 1$ ).

### 3.4.3 The flow on the invariant set $(\partial\Sigma)_2$

The dynamics in the invariant set  $(\partial\Sigma)_2$  is governed by the differential equations:



$$x_1' = (x_1^2 - 1) \left( x_1 + \frac{\sqrt{6}}{6} \frac{\partial_\phi V(\phi)}{V(\phi)} \right), \quad (3.65)$$

and (3.58)

In the invariant set  $(\partial\Sigma)_2$  there are two invariant subsets: the set

$$W_1 = \{p = (\phi, x_1, x_2, x_3) \in \Sigma : |x_1| = 1\},$$

and the (possibly empty) class,  $P$ , of singular points  $\mathbf{p}$  with coordinates  $\phi = \phi_2$  with  $\chi(\phi_2) \neq 0$ ,  $V'(\phi_2) = 0$ , and  $x_1 = 0$ , hence  $x_3 = \sqrt{\frac{3}{V(\phi_2)}}$ . We assume that  $V$  has only isolated singular points. Each singular point  $\mathbf{p} \in P$  is

1. a saddle if  $V''(\phi_2) < 0$ ,
2. a stable node if  $0 < V''(\phi_2) \leq \frac{3}{4}V(\phi_2)$ , and
3. a stable focus if  $V''(\phi_2) > \frac{3}{4}V(\phi_2)$ .

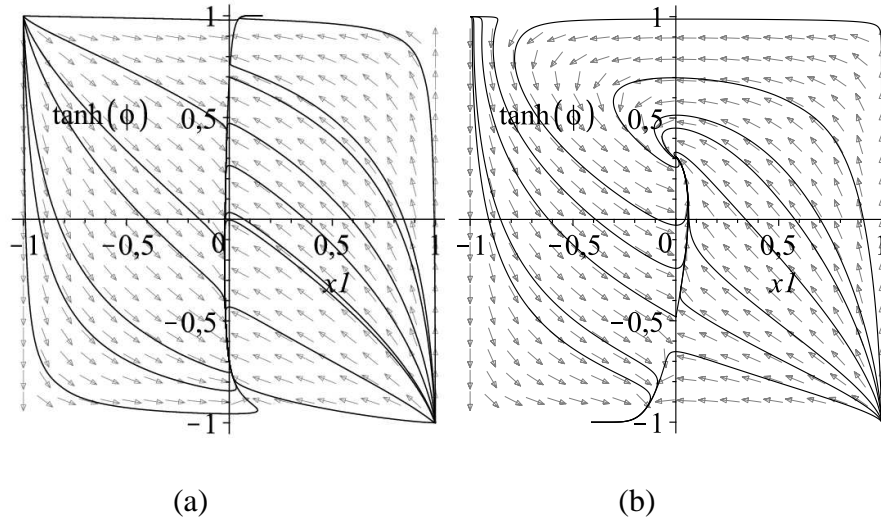


Figure 3.2: Phase plane  $(x_1, \varphi)$  for the model with potential (3.66) (a) for  $A = 3.25$ ,  $B = -2$ , and  $\mu = 0.5$  the singular points of the system are a saddle located at  $(x_1, \varphi) = (0, 0.6993)$  and a node at  $(x_1, \varphi) = (0, -0.6993)$ . (b) for  $A = 0.62$ ,  $B = 0.79$ , and  $\mu = -1.12$  the singular points of the system are a saddle located at  $(x_1, \varphi) = (0, -0.4806)$  and an stable spiral at  $(x_1, \varphi) = (0, 0.3078)$ .

### 3.4.3.1 The Albrecht-Skordis potential

Albrecht and Skordis [412] have proposed a particularly attractive model of quintessence. It is driven by a potential which introduces a small minimum to the exponential potential:

$$V(\phi) = e^{-\mu\phi} (A + (\phi - B)^2). \quad (3.66)$$

Unlike previous quintessence models, late-time acceleration is achieved without fine tuning of the initial conditions. The authors argue that such potentials arise naturally in the low-energy limit of  $M$ -theory. The constant parameters,  $A$  and  $B$ , in the potential take values of order 1 in Planck units, so there is also no fine tuning of the potential (we suppose also that  $\mu \neq 0$ ). They show that, regardless of the initial conditions,  $\rho_\phi$  scales, with  $\rho \propto \rho_\phi \propto t^{-2}$  during the radiation and matter eras, but leads to permanent vacuum domination and accelerated expansion after a time which can be close to the present.

The extremes of the potential (3.66) are located at  $\phi^\pm = \frac{1+B\mu \pm \sqrt{1-A\mu^2}}{\mu}$ . They are real if  $1 \geq \mu^2 A$ . The local minimum (respectively, local maximum) is located at  $\phi^-$  (respectively  $\phi^+$ ) since

$$\pm V''(\phi^\pm) = -2V_0 \sqrt{1-A\mu^2} e^{-(1+B\mu \pm \sqrt{1-A\mu^2})} < 0.$$

By using the formalism developed here we find that the singular point associated to  $\phi^+$  is always a saddle point of the corresponding phase portrait (see figures 3.2 (a)-b). The singular point associated to  $\phi^-$  could be either an stable node (see 3.2 (a)) or an stable spiral (see 3.2 (b)) if

$$\frac{8(3+2\mu^2)}{(3+4\mu^2)^2} < A < \frac{1}{\mu^2}$$

or

$$A < \frac{8(3+2\mu^2)}{(3+4\mu^2)^2}.$$

As before we have introduced a new scalar field  $\varphi = \tanh(\phi)$  in order to make the numerics. Observe that, almost all the initial points in the phase plane past asymptote (in figures 3.2 (a)-(b)) to the points  $p_{1,2}$  with coordinates  $(y, \varphi) = (\mp 1, \pm 1)$  (they are associated respectively with the infinite boundaries  $\phi = \pm\infty$ ). In the case (a) there are some orbits past asymptotic to  $P_{1,2}$  converging to the point  $P_4$  with coordinates  $(x_1, \varphi) = (\mu/\sqrt{6}, +1)$  which is a singular point located at the region  $\phi = \infty$ , in this example, its center manifold acts as an exponential attractor (for more details see the next section) whereas the singular point with coordinates  $(x_1, \varphi) = (\mu/\sqrt{6}, -1)$  acts as a saddle. In the case (b) we have a similar situation. Actually, there exist two singular points with coordinates  $x_1 = \mu/\sqrt{6}$  each one contained in the invariant manifolds  $\varphi = \pm 1$  (i.e.,  $\phi = \pm\infty$ ) respectively. Each one has a center manifold which acts as an exponential

attractor for nearby orbits.

These examples suggest the divergence of  $\phi$  towards the past. We proceed to prove the following

**Theorem 22** *Let  $\chi(\phi)$  and  $V(\phi)$  be positive functions of class  $C^3$ , such that  $\chi$  satisfies that  $0 < \chi(\phi) < +\infty$  for any compact set, and has, at most, a finite number of isolated singular points. Let  $\gamma \in (0, \frac{4}{3}) \cup (\frac{4}{3}, 2)$ . Let  $\mathbf{p}$  a point in  $\Sigma$ , and  $O^-(\mathbf{p})$  the past orbit of  $\mathbf{p}$  under the flow  $\mathbf{g}^\tau$  of (3.23)-(3.26) in  $\Sigma$ . Then,  $\phi$  is unbounded in  $O^-(\mathbf{p})$  for almost all  $\mathbf{p}$ .*

**Proof.** For the demonstration we will consider only interior points of  $\Sigma$ , since its boundary  $\partial\Sigma = (\partial\Sigma)_1 \cup (\partial\Sigma)_2$  is of dimension  $2 < \dim\Sigma$ . Form this fact follows that this set is of zero Lebesgue measure.<sup>6</sup>

Let us consider the time reversal transformation  $(\tau, \phi, x_1, x_2, x_3) \rightarrow (-\tau, \phi, x_1, x_2, x_3)$ . Using this transformation the system becomes

$$\phi' = -\sqrt{\frac{2}{3}}x_1, \quad (3.67)$$

$$x_1' = -x_1^3 - \frac{1}{2}(x_2^2\gamma - 2)x_1 + \frac{x_3^2}{3\sqrt{6}}\frac{dV(\phi)}{d\phi} - \frac{(4-3\gamma)x_2^2}{2\sqrt{6}}\frac{d\ln\chi(\phi)}{d\phi}, \quad (3.68)$$

$$x_2' = -\frac{1}{2}x_2(2x_1^2 + (x_2^2 - 1)\gamma) + \frac{(4-3\gamma)x_1x_2}{2\sqrt{6}}\frac{d\ln\chi(\phi)}{d\phi}, \quad (3.69)$$

$$x_3' = -\frac{1}{2}x_3(2x_1^2 + x_2^2\gamma), \quad (3.70)$$

where new the comma denotes derivative with respect to  $-\tau$ .

Let  $\mathbf{p}_0 = (\phi_0, x_{10}, x_{20}, x_{30}) \in \text{Int}\Sigma$  such that exists  $K, |\phi| < K$ , for all  $\mathbf{p} = (\phi, x_1, x_2, x_3) \in O^+(\mathbf{p}_0)$ , where  $O^+(\mathbf{p}_0)$  denotes the future orbit of (3.67)-(3.70) passing through  $\mathbf{p}_0$ . Then, the following inequalities hold

$$-1 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq \sqrt{\frac{3}{\inf_{\phi \in [-K, K]} V(\phi)}}, \forall \mathbf{p} \in O^+(\mathbf{p}_0). \quad (3.71)$$

Since  $V(\phi) > 0$  follows that  $O^+(\mathbf{p}_0)$  is contained in a compact subset of  $\Sigma$ . This implies that there exists a non empty, closed, connected invariant manifold  $\omega(\mathbf{p}), \forall \mathbf{p} \in O^+(\mathbf{p}_0)$ .

Since  $\gamma > 0$ , follows that the function

$$Z : \text{Int}\Sigma \rightarrow (0, \infty), p = (\phi, x_1, x_2, x_3) \rightarrow Z(p) = \left(\frac{x_3}{x_2}\right)^2 \chi(\phi)^{-2+\frac{3\gamma}{2}} \quad (3.72)$$

---

<sup>6</sup>However, in the previous sections we have presented several numerical examples concerning the dynamics on  $(\partial\Sigma)_1$  and on  $(\partial\Sigma)_2$ , which supported the unboundedness of the scalar field towards the past.

satisfies  $Z' = \nabla Z \cdot (\phi', x'_1, x'_2, x'_3) = -\gamma Z$ . This means that  $Z$  is a monotonic function for the flow of (3.67)-(3.70), defined in the invariant set  $Int\Sigma$ . Since  $\chi$  is a positive function of class  $C^3$  follows that  $Z$  is  $C^3$  in  $Int\Sigma$ .

The function  $Z$  takes values in the interval  $(0, +\infty)$ . From the hypothesis about  $\chi$  follows that it cannot not be zero or infinite unless  $|\phi| \rightarrow \infty$ . Thus, by construction follows that  $Z(\mathbf{p}) \rightarrow 0$  if and only if  $\mathbf{p} \rightarrow \mathbf{q}$  with  $\mathbf{q}$  such that  $x_3 = 0$  and  $Z(\mathbf{p}) \rightarrow +\infty$  if and only if  $\mathbf{p} \rightarrow \mathbf{q}$  with  $\mathbf{q}$  such that  $x_2 = 0$ .

If  $x_2 \rightarrow 0$  and  $x_3 \rightarrow 0$  simultaneously, then by the definition of  $\Sigma$  follows that  $x_1 \rightarrow \pm 1$ , and from (3.67) follows that  $\phi \rightarrow \mp\infty$ , which contradicts that  $\phi$  is bounded at  $O^+(\mathbf{p}_0)$ . This implies that  $x_2$  and  $x_3$  cannot simultaneously tend to zero in  $\omega(\mathbf{p})$  for all  $\mathbf{p} \in O^+(\mathbf{p}_0)$ .

Applying the Monotonicity Principle (theorem 18) follows that  $\omega(\mathbf{p}) \subset \{\mathbf{p} \in \partial\Sigma : x_2 \neq 0\} \cap \{\mathbf{p} \in \Sigma : |\phi| < K\} = \{\mathbf{p} \in \Sigma : x_2 > 0, x_3 = 0\} \cap \{\mathbf{p} \in \Sigma : |\phi| < K\} \subset (\partial\Sigma)_1 \setminus W_1$ .

Let us define  $V := \{\mathbf{p} \in \Sigma : x_2 > 0, x_3 = 0\} \cap \{\mathbf{p} \in \Sigma : |\phi| < K\} = \{(\phi, x_1, x_2) \in \mathbb{H}^3 : |\phi| < K, x_1^2 + x_2^2 = 1\}$ .

Let  $\mathbf{q}_0 \in \omega(\mathbf{p}_0)$ . From the invariance of  $\omega(\mathbf{p}_0)$  follows that the solution  $\mathbf{x}(\tau, \mathbf{q}_0) \in \omega(\mathbf{p}_0)$ .

Taking the local chart  $(V, g|_V)$  with  $g$  defined by  $g : (\partial\Sigma)_1 \setminus W_1 \rightarrow \mathbb{R} \times (-1, 1)$ ,  $(\phi, x_1, x_2) \rightarrow (\phi, x_1) = (\xi_1, \xi_2)$ , follows that flow of (3.67)-(3.70) is topologically equivalent in a neighborhood of  $\mathbf{q}_0$  to the flow of

$$\xi'_1 = f_1(\xi_1, \xi_2) = -\sqrt{\frac{2}{3}}\xi_2, \quad (3.73)$$

$$\xi'_2 = f_2(\xi_1, \xi_2) = \frac{1}{2}(1 - \xi_2^2) \left( (2 - \gamma)\xi_2 - \frac{4 - 3\gamma}{\sqrt{6}} \frac{d \ln \chi(\xi_1)}{d\xi_1} \right), \quad (3.74)$$

defined in a neighborhood of  $\xi_0 = h(\mathbf{q}_0)$ .

From the topological equivalence between flows follows that there exists a non-empty  $\omega$ -limit set of  $\xi_0$ .

Let  $S = h(V) := \mathbb{R} \times (-1, 1)$ .  $S$  is an open simply connected set of  $\mathbb{R}^2$ . The question is: What invariant closed subsets of  $\bar{S} = \mathbb{R} \times [-1, 1]$  can be candidates to  $\omega(\xi_0)$ ?

The subsets of  $\bar{S}$  with  $\xi_2 = \pm 1$  are discarded since on them  $\phi$  is unbounded.

Let  $L \subset \bar{S}$  be a closed positively invariant set of (3.73)-(3.74). From the Dulac's criterion (theorem 21) follows that do not exists periodic orbits in  $\bar{S}$ , since the function

$$B : \mathbb{R} \times (-1, 1) \rightarrow \mathbb{R}, \xi \rightarrow B(\xi) = (1 - \xi_2^2)^{-1}. \quad (3.75)$$

satisfies  $B \in C^1(\mathbb{R} \times (-1, 1))$  and  $\nabla \cdot (Bf) = \frac{\partial}{\partial \xi_1} Bf_1 + \frac{\partial}{\partial \xi_2} Bf_2 = 1 - \frac{\gamma}{2} > 0$ . Thus, it

is a Dulac's function in  $S$ . Besides,  $\partial S = ([-K, K] \times \{-1, 1\}) \cup (\{-K, K\} \times (-1, 1))$  is not a closed orbit. Hence  $L \subset \bar{S}$  do not contain periodic orbits.

From the corollary of Poincaré-Bendixon Theorem 20, follows that the only possible invariant sets are the singular points with  $\xi_1$  bounded, or heteroclinic sequences joining them.

The system (3.73)-(3.74) admits a (possibly empty) family of singular points with  $\xi_1$  bounded  $Q := \{\mathbf{q} = (q_1, 0) \in \bar{S} : \chi'(q_1) = 0\}$ .

Suppose that  $Q = \emptyset$ , i.e.,  $\chi'(q_1) \neq 0, \forall q_1, |q_1| < K$ . In this case the orbit  $O^+(\xi_0)$  tends to a point with  $\xi_1 = \pm 1$ , following the unboundedness of  $\phi$ . We conclude that there are not interior points of  $\Sigma$  leading to a bounded past orbit.

Suppose that  $Q \neq \emptyset$ . Let  $\mathbf{q} \in Q$ . The eigenvalues of the Jacobian matrix,  $\frac{\partial f^i}{\partial \xi_j}(\mathbf{q}), i, j = 1, 2$ , are  $\mu^\pm = \frac{2-\gamma}{4} \pm \sqrt{\left(\frac{2-\gamma}{4}\right)^2 + \frac{4-3\gamma}{6} \frac{\chi''(q_1)}{\chi(q_1)}}$ .

Denoting the sets  $Q^\pm = \{\mathbf{q} \in Q : \pm \chi''(q_1) > 0\}$  y  $Q^0 = \{\mathbf{q} \in Q : \chi(q_1) = 0\}$ . At least one of these sets is nonempty. Let us define  $R = \{\mathbf{p} \in [-K, K] \times [-1, 1] : \lim_{\tau \rightarrow \infty} \mathbf{g}^\tau(\mathbf{p}) = \mathbf{q}\}$ . Are distinguished five cases <sup>7</sup>:

- $\mathbf{q} \in Q^-, 0 < \gamma < \frac{4}{3}$ , or  $\mathbf{q} \in Q^+, \frac{4}{3} < \gamma < 2$ , then  $\mathcal{E}^u(\mathbf{q})$  is 2D, hence  $R = \emptyset$ .
- $\mathbf{q} \in Q^-, \frac{4}{3} < \gamma < 2$ , or  $\mathbf{q} \in Q^+, 0 < \gamma < \frac{4}{3}$ , then  $\mathcal{E}^u(\mathbf{q})$  is 1D and  $\mathcal{E}^s(\mathbf{q})$  is 1D is such way that  $R \subset N, \text{leb}(N) = 0$ .
- $\mathbf{q} \in Q^0$ , then  $\mathcal{E}^c(\mathbf{q})$  is 1D and  $\mathcal{E}^u(\mathbf{q})$  is 1D, hence,  $R \subset \mathcal{E}^c(\mathbf{q}), \text{leb}(\mathcal{E}^c(\mathbf{q})) = 0$ .

Therefore, all solutions future asymptotic to  $\mathbf{q}$  (and then with  $\phi$  bounded towards the future) must lie on an stable manifold or center manifold of dimension  $r < 2$ , and then contained in a subset of  $[-K, K] \times [-1, 1]$  with zero Lebesgue measure. Since there are at most a finite number of such  $q$  the result of the theorem follows. ■

#### 3.4.4 The flow in the invariant set $\rho_r = 0$ as $\phi \rightarrow +\infty$

As we commented before, in order to investigate the generic asymptotic behavior of the system (3.23)-(3.26) restricted to  $\Sigma$  it is sufficient to study the region where  $\phi = \pm\infty$ . Let us remark, however, the region  $\phi = \pm\infty$  is not exclusively associated to the asymptotic behavior to the past. As has been investigated in section 3.3 (see also the degree thesis [413]):

- If  $V \geq 0$  and  $V(\phi) = 0 \Leftrightarrow \phi = 0$ , where  $\phi = 0$  is a local minimum of the potential;
- If there exists  $A$  such that  $V$  bounded in  $A$  implies  $V'(\phi)$  is bounded in  $A$ ;

<sup>7</sup>We denote the stable, unstable and center manifolds of  $q$  by  $\mathcal{E}^s(\mathbf{q}), \mathcal{E}^u(\mathbf{q})$  and  $\mathcal{E}^c(\mathbf{q})$  respectively. By  $\text{leb}(A)$  we denote de Lebesgue measure of  $A \subset \mathbb{R}^2$ .

iii) If there exists some constant  $K$  (either positive or negative) such that

$$\chi'(\phi)/\chi(\phi) \leq 2K/(2 - \gamma)(4 - 3\gamma);$$

iv) If  $V'(\phi) > 0$  for  $\phi > 0$  and  $V'(\phi) < 0$  for  $\phi < 0$ ; then

$$\lim_{t \rightarrow \infty} \rho = 0 = \lim_{t \rightarrow \infty} \dot{\phi}$$

and  $\lim_{t \rightarrow \infty} \phi$  exists and it is equal to  $+\infty$ ,  $0$  or  $-\infty$ .

The case  $\lim_{t \rightarrow \infty} \phi = 0$  holds only if  $\lim_{t \rightarrow \infty} H(t) = 0$ , which can be achieved, for instance, if  $3H(t_0)^2 < \min \{ \lim_{\phi \rightarrow \infty} V(\phi), \lim_{\phi \rightarrow -\infty} V(\phi) \}$ .

As discussed in section 3.3 these results are extensions to the non-minimal coupling context of the results proved in [404] (see Propositions 2 and 3 of that reference).

In this section we follow the nomenclature and formalism introduced in [403].

**Definition 25 (Function well-behaved at infinity [403])** *Let  $V : \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^2$  non-negative function. Let there exist some  $\phi_0 > 0$  for which  $V(\phi) > 0$  for all  $\phi > \phi_0$  and some number  $N$  such that the function  $W_V : [\phi_0, \infty) \rightarrow \mathbb{R}$ ,*

$$W_V(\phi) = \frac{V'(\phi)}{V(\phi)} - N$$

*satisfies*

$$\lim_{\phi \rightarrow \infty} W_V(\phi) = 0. \quad (3.76)$$

*Then we say that  $V$  is Well Behaved at Infinity (WBI) of exponential order  $N$ .*

It is important to point out that  $N$  may be  $0$ , or even negative. Indeed the class of WBI functions of order  $0$  is of particular interest, containing all non-negative polynomials as remarked in [403].

**Theorem 23 (Theorem 2, [403])** *Let  $V$  be a WBI function of exponential order  $N$ , then, for all  $\lambda > N$ ,*

$$\lim_{\phi \rightarrow +\infty} e^{-\lambda\phi} V(\phi) = 0.$$

In order to classify the smoothness of WBI functions at infinity it is introduced the definition

**Definition 26** *Let be some coordinate transformation  $\varphi = f(\phi)$  mapping a neighborhood of infinity to a neighborhood of the origin. If  $g$  is a function of  $\phi$ ,  $\bar{g}$  is the function of  $\varphi$  whose domain is the range of  $f$  plus the origin, which takes the values;*

$$\bar{g}(\varphi) = \begin{cases} g(f^{-1}(\varphi)) & , \quad \varphi > 0 \\ \lim_{\phi \rightarrow \infty} g(\phi) & , \quad \varphi = 0 \end{cases}$$

Table 3.1: Simple examples of WBI behavior at large  $\phi$ .  $n$  and  $\lambda$  are arbitrary constants. Adapted from [403].

$V(\phi)$	$W_V(\phi)$	$\varphi = f(\phi)$	$\overline{W}_V(\varphi)$	$\overline{f}'(\varphi)$
$\left \frac{\lambda}{n}\right  \phi^n$	$n\phi^{-1}$	$\phi^{-\frac{1}{2}}$	$n\varphi^2$	$-\frac{1}{2}\varphi^3$
$e^{\lambda\phi}$	0	$\phi^{-1}$	0	$-\varphi^2$
$2e^{\lambda\sqrt{\phi}}$	$\lambda\phi^{-\frac{1}{2}}$	$\phi^{-\frac{1}{4}}$	$\lambda\varphi^2$	$-\frac{1}{4}\varphi^5$
$(A + (\phi - B)^2) e^{-\mu\phi}$	$\frac{2(\phi-B)}{A+(B-\phi)^2}$	$\phi^{-\frac{1}{2}}$	$-\frac{2\varphi^2(B\varphi^2-1)}{A\varphi^4+(B\varphi^2-1)^2}$	$-\frac{1}{2}\varphi^3$
$\left(1 - e^{-\lambda^2\phi}\right)^2$	$-\frac{2\lambda^2}{1-e^{\lambda^2\phi}}$	$\phi^{-1}$	$-\frac{2\lambda^2}{1-\frac{e}{\varphi}}$	$-\varphi^2$
$\ln \phi$	$(\phi \ln \phi)^{-1}$	$(\ln \phi)^{-1}$	$\varphi e^{-\frac{1}{\varphi}}$	$-\varphi e^{-\frac{2}{\varphi}}$
$\phi^2 \ln \phi$	$2\phi^{-1} + (\phi \ln \phi)^{-1}$	$(\ln \phi)^{-1}$	$(2 + \varphi)e^{-\frac{1}{\varphi}}$	$-\varphi e^{-\frac{2}{\varphi}}$

**Definition 27 (Class k WBI functions [403])** A  $C^k$  function  $V$  is class  $k$  WBI if it is WBI and if there exists  $\phi_0 > 0$  and a coordinate transformation  $\varphi = f(\phi)$  which maps the interval  $[\phi_0, \infty)$  onto  $(0, \epsilon]$ , where  $\epsilon = f(\phi_0)$  and  $\lim_{\phi \rightarrow \infty} f = 0$ , with the following additional properties:

- i)  $f$  is  $C^{k+1}$  and strictly decreasing.
- ii) the functions  $\overline{W}_V(\varphi)$  and  $\overline{f}'(\varphi)$  are  $C^k$  on the closed interval  $[0, \epsilon]$ .
- iii)  $\frac{d\overline{W}_V}{d\varphi}(0) = \frac{d\overline{f}'}{d\varphi}(0) = 0$ .

We designate the set of all class  $k$  WBI functions  $\mathcal{E}_+^k$ . In table 3.1 are displayed simple examples of WBI behavior at large  $\phi$ .

Let be  $V, \chi \in \mathcal{E}_+^2$  with exponential orders  $N$  and  $M$ , respectively. Let be the set

$$\Sigma_\epsilon = \left\{ \mathbf{p} \in (\phi, x_1, x_2, x_3) \in \Sigma \mid \phi > \epsilon^{-1} \right\},$$

where  $\epsilon$  is any positive constant which is chosen sufficiently small so as to avoid any points where  $V$  or  $\chi = 0$ , thereby ensuring that  $\overline{W}_V(\varphi)$  and  $\overline{W}_\chi(\varphi)$  are well-defined.

Let be defined the coordinate transformation

$$(\phi, x_1, x_2, x_3) \xrightarrow{\varphi=f(\phi)} (\varphi, x_1, x_2, x_3) \quad (3.77)$$

over  $\Sigma_\epsilon$ , where  $f(\phi)$  satisfy the conditions in definition 27 for  $k = 2$ .

Taking the coordinate transformation

$$(\phi, x_1, x_2, x_3) \xrightarrow{\varphi=f(\phi)} (\varphi, x_1, x_2, x_3) \xrightarrow{g} (\varphi, x_1, x_2),$$

where  $g$  is a homeomorphism with inverse

$$(\varphi, x_1, x_2) \xrightarrow{g^{-1}} \left( \varphi, x_1, x_2, \sqrt{\frac{3(1 - x_2^2 - x_3^2)}{\bar{V}(\varphi)}} \right),$$

we obtain the 3-dimensional dynamical system:

$$\varphi' = \sqrt{\frac{2}{3}} \bar{f}'(\varphi) x_1. \quad (3.78)$$

$$\begin{aligned} x_1' = x_1^3 + \frac{1}{2} (x_2^2 \gamma - 2) x_1 - \frac{(1 - x_1^2 - x_2^2)}{\sqrt{6}} (\bar{W}_V(\varphi) + N) + \\ + \frac{x_2^2(4 - 3\gamma)}{2\sqrt{6}} (\bar{W}_\chi(\varphi) + M), \end{aligned} \quad (3.79)$$

$$x_2' = \frac{1}{2} x_2 (2x_1^2 + (x_2^2 - 1) \gamma) + \frac{x_1 x_2 (-4 + 3\gamma)}{2\sqrt{6}} (\bar{W}_\chi(\varphi) + M). \quad (3.80)$$

We may identify  $\Sigma_\epsilon$  with its projection into  $\mathbb{R}^3$  so that we have  $\Sigma_\epsilon = \{0 < \varphi < f(\epsilon^{-1}), 0 < x_1^2 + x_2^2 < 1\}$ . The variable  $x_3$  can be treated as a function on  $\Sigma_\epsilon$  defined by the constraint equation which becomes

$$x_1^2 + x_2^2 + \frac{1}{3} x_3^2 \bar{V}(\varphi) = 1. \quad (3.81)$$

Since  $\bar{f}'$ ,  $\bar{W}_V$  and  $\bar{W}_\chi$  are  $C^2$  at  $\varphi = 0$  we may extend (3.78)-(3.80) onto the boundary of  $\Sigma_\epsilon$  to obtain a  $C^2$  system on the closure of  $\Sigma_\epsilon$ , i.e.,  $\bar{\Sigma}_\epsilon$ . From definition 27,  $\bar{f}'$ ,  $\bar{W}_V$  and  $\bar{W}_\chi$  vanish at the origin and are each of second order or higher in  $\varphi$  and  $\bar{f}'$  is negative on  $\Sigma_\epsilon$ .

#### 3.4.4.1 Location, existence and stability conditions of the singular points. Cosmological parameters

The system (3.78)-(3.80) admits the singular points at infinity (i.e., with  $\phi$  unbounded) labelled by  $P_i, i \in \{1, 2, 3, 4, 5, 6\}$ . In the following we discuss the existence and the stability conditions for the singular points. In the table 3.2 are displayed the values of some cosmological magnitudes of interest for the singular points (the deceleration parameter, the effective EoS parameter for the total matter, etc).<sup>8</sup>

1. The singular point  $P_1$  with coordinates  $\varphi = 0$ ,  $x_1 = -1$ , and  $x_2 = 0$  exists for all the values of the free parameters. The eigenvalues of the linearized system around  $P_1$  are  $\lambda_{1,1} = 2 - \sqrt{2/3}N$ ,  $\lambda_{1,2} = \frac{2-\gamma}{2} - \frac{M(-4+3\gamma)}{2\sqrt{6}}$  and  $\lambda_{1,3} = 0$ . Hence the singular

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<sup>8</sup>For some specific examples there exist singular points of (3.78)-(3.80) with  $\varphi > 0$ . We omit the description of them because they do not correspond to the limit  $|\phi| \rightarrow \infty$ . However, as we will see later, some of these points can attract orbits located initially at the “infinity boundary”.



Table 3.2: The properties of the singular points for the system (3.78)-(3.80). We use the notations  $\alpha = 3(N(\gamma - 2) + M(3\gamma - 4))$ ,  $\beta = 2(2N - M(3\gamma - 4))$ ,  $\delta = \frac{M(3\gamma - 4)}{\sqrt{6}(\gamma - 2)}$ , and  $\Gamma = \frac{\sqrt{2(\gamma-2)(3\gamma-2)}}{4-3\gamma}$ .

Point	$x_1$	$x_2$	$\Omega_{de}$	$w_{tot}$	Acceleration?
$P_1$	-1	0	1	1	no
$P_2$	1	0	1	1	no
$P_3$	$\delta$	$\sqrt{1 - \delta^2}$	$\delta^2$	$\gamma + (\gamma - 1)\delta$	$0 < \gamma < \frac{2}{3}$ and $ M  < \Gamma$
$P_4$	$-\frac{N}{\sqrt{6}}$	0	1	$-1 + \frac{N^2}{3}$	$N^2 < 2$
$P_{5,6}$	$-\frac{6\sqrt{6}\gamma}{\beta}$	$\mp \sqrt{\frac{2\beta(2\alpha+\beta)-432\gamma}{\beta}}$	$-\frac{2(2\alpha+\beta)}{\beta\gamma} + \frac{432\gamma}{\beta^2} + 1$	$\frac{(\gamma+2)\beta^2+4\alpha(\gamma+1)\beta-432\gamma^2}{\beta^2\gamma}$	$\frac{\alpha}{\beta} < -\frac{1}{3}$

point is nonhyperbolic, then, the Hartman-Grobman theorem does not applies. By the Center-manifold theorem there exist:

- (a) an stable invariant subspace of dimension two (tangent to the  $x_1$ - $x_2$  plane) if: i) the potential is a WBI function of exponential order  $N > \sqrt{6}$  and the coupling function is a WBI function of exponential order  $M < -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$  (provided  $0 < \gamma < \frac{4}{3}$ ), or ii) the barotropic index satisfies  $\frac{4}{3} < \gamma < 2$ , the potential is a WBI function of exponential order  $N > \sqrt{6}$  and the coupling function is a WBI function of exponential order  $M > -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$ ;
  - (b) an unstable invariant subspace of dimension two (tangent to the  $x_1$ - $x_2$  plane) provided the potential is a WBI function of exponential order  $N < \sqrt{6}$  and the coupling function is a WBI function of exponential order  $M$  such that  $M > -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$  (respectively,  $M < -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$ ) provided  $0 < \gamma < \frac{4}{3}$  (respectively,  $\frac{4}{3} < \gamma < 2$ );
  - (c) a 1-dimensional center manifold which is tangent to the singular point in the direction of the axis  $\varphi$ . This center manifold can be 2-dimensional or even 3-dimensional (see the discussion on the point 3).
2. The singular point  $P_2$  with coordinates  $\varphi = 0$ ,  $x_1 = 1$ , and  $x_2 = 0$  exists for all the values of the free parameters. The eigenvalues of the linearized system around  $P_2$  are  $\lambda_{2,1} = 2 + \sqrt{2/3}N$ ,  $\lambda_{2,2} = \lambda_{1,2}$  and  $\lambda_{2,3} = 0$  (see point 1). Hence the singular point is nonhyperbolic, then, the Hartman-Grobman theorem does not applies. By the Center-manifold theorem there exist:

- (a) an stable invariant subspace of dimension two (tangent to the  $x_1$ - $x_2$  plane) if:
  - i)  $N < -\sqrt{6}$ ,  $M > \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$  for  $0 < \gamma < \frac{4}{3}$ , or ii)  $\frac{4}{3} < \gamma < 2$ ,  $N < -\sqrt{6}$  and  $M < \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$ ;
- (b) an unstable invariant subspace of dimension two (tangent to the  $x_1$ - $x_2$  plane) provided  $N > -\sqrt{6}$ , and  $M$  such that  $M < \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$  (respectively  $M > \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$ ) provided  $0 < \gamma < \frac{4}{3}$  (respectively  $\frac{4}{3} < \gamma < 2$ );
- (c) a 1-dimensional center manifold which is tangent to the singular point in the direction of the axis  $\varphi$ . This center manifold can be 2-dimensional or even 3-dimensional (see the discussion on the point 3).

In the following section we shall study the initial spacetime (big bang) singularity. The singular points  $P_{1,2}$  can account for that singularity. They are in the same phase portrait for the values  $-\sqrt{6} < N < \sqrt{6}$  and  $-\frac{\sqrt{6}(\gamma-2)}{3\gamma-4} < M < \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$  and  $0 < \gamma < \frac{4}{3}$  (in which case they have a 2-dimensional unstable manifold and a 1-dimensional center center respectively). It is easy to show that the Hubble parameter (and the matter density) of the cosmological solutions associated to these points diverges into the past. The scalar field also diverges, it equals to  $+\infty$  (respectively  $-\infty$ ) for  $P_1$  (respectively  $P_2$ ). However, even in this case, the past attractor corresponds to  $P_1$  since  $\overline{f'} < 0$  and for  $x_1 > 0$  the orbits enter the phase portrait and  $P_2$  acts as a saddle. The last point can be a past attractor only on a set of measure zero (if  $\varphi = 0$ ).

3. The singular point  $P_3$  with coordinates  $\varphi = 0$ ,  $x_1 = \frac{M(-4+3\gamma)}{\sqrt{6}(-2+\gamma)}$ , and  $x_2 = \sqrt{1 - \frac{M^2(4-3\gamma)^2}{6(-2+\gamma)^2}}$  exists if  $0 < \gamma < \frac{4}{3}$  and  $-\frac{\sqrt{6}(-2+\gamma)}{-4+3\gamma} \leq M \leq \frac{\sqrt{6}(-2+\gamma)}{-4+3\gamma}$ . The eigenvalues of the matrix of derivatives evaluated at the singular point are  $\lambda_{3,1} = \frac{6(\gamma-2)^2 - M^2(4-3\gamma)^2}{12(\gamma-2)}$ ,  $\lambda_{3,2} = -\frac{3\gamma M^2}{2} + (M + N)M + \frac{2(N-M)M}{3(\gamma-2)} + \gamma$ , and  $\lambda_{3,3} = 0$ . Hence the singular point is nonhyperbolic, then, the Hartman-Grobman theorem does not applies. Under the above existence conditions we find, by the Center Manifold Theorem, that there exists an stable manifold of dimension two for the values of the parameters: i)  $M < 0$  and  $N > \frac{M^2(4-3\gamma)^2 - 6(\gamma-2)\gamma}{2M(3\gamma-4)}$  or ii)  $M > 0$  and  $N < \frac{M^2(4-3\gamma)^2 - 6(\gamma-2)\gamma}{2M(3\gamma-4)}$ . Otherwise there exist an unstable manifold of dimension one (in this case the stable subspace is 1-dimensional). The center manifold is in both cases 1-dimensional. If  $M = \mp \frac{\sqrt{6}(-2+\gamma)}{-4+3\gamma}$  this singular point reduces to  $P_{1,2}$ . In this case the center subspace is the 2-dimensional  $\varphi$ - $x_2$  plane. The center manifold is tangent to the center subspace at the singular point. If additionally  $|N| = \sqrt{6}$ , the center manifold is 3-dimensional.
4. The singular point  $P_4$  with coordinates  $\varphi = 0$ ,  $x_1 = -\frac{N}{\sqrt{6}}$ ,  $x_2 = 0$  exists if  $|N| \leq \sqrt{6}$ . Observe that this point reduces to  $P_{1,2}$  if  $N^2 = 6$ . The eigenvalues of the

matrix of derivatives evaluated at the singular point has the eigenvalues  $\lambda_{4,1} = \frac{1}{6}(N^2 - 6) \leq 0$ ,  $\lambda_{4,2} = \frac{1}{6}N(2M + N) - \frac{1}{4}(MN + 2)\gamma$  and  $\lambda_{4,3} = 0$ . Hence the singular point is nonhyperbolic and, as before, the Hartman-Grobman theorem does not apply. However, we can use the Center Manifold Theorem to investigate the stability of this singular point. The structure of the center manifold is as follows:

- (a) if  $\lambda_{4,1} < 0$  and  $\lambda_{4,2} \neq 0$  the center manifold is tangent to the  $\varphi$ -axis. Then, it is 1-dimensional. Before analyzing this case in detail, we will provide additional information about the structure of the center manifold;
- (b) if  $M = \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)}$  and  $N^2 < 6$ , the center manifold is tangent to the  $\varphi$ - $x_2$  plane;
- (c) if  $N^2 = 6$  and  $M \neq \mp \frac{\sqrt{6}(-2+\gamma)}{-4+3\gamma}$ , the center manifold is tangent to the  $\varphi$ - $x_1$  plane;
- (d) if  $N^2 = 6$  and  $M = \mp \frac{\sqrt{6}(-2+\gamma)}{-4+3\gamma}$ , the center manifold is 3-dimensional.

The local behavior described in the cases above (excluding the first case) is in some way special. It requires fine tuning of the free parameter. However, the typical behavior is the existence of a one dimensional center manifold  $C_N$  through  $P_4$ , which is tangent to the  $x_2$ -axis (if  $\lambda_{4,1} < 0$  and  $\lambda_{4,2} \neq 0$ ). In accordance with our previous comments on the properties of center manifolds it is clear that  $C_N$  is an exponential attractor on a sufficiently small neighborhood of  $P_4$  and it is intuitively obvious from the geometry (for instance, observe that in the figure 3.5 there exists a line passing through  $P_4$  which is an exponential attractor) that any solutions past asymptotic to  $P_4$  must lie on the center manifold.

Let us investigate the case in which  $\lambda_{4,1} < 0$  and  $\lambda_{4,2} \neq 0$ . Of course, in this case the stable manifold is at least 1-dimensional (and as we mentioned before, the center manifold is 1-dimensional).

The structure of the stable subspace is as follows:

- (a) if the potential is of exponential order zero ( $N = 0$ ), then, the singular point has coordinates  $(0, 0, 0)$ . The eigenvalues of the linearization are  $(-1, 0, -\frac{\gamma}{2})$  and in this case, the stable subspace is tangent to the  $x_1$ - $x_2$  plane;
- (b) if  $0 < \gamma < \frac{4}{3}$ ,  $-\sqrt{6} < N < 0$ , and  $M > \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)}$ ; or
- (c) if  $\frac{4}{3} < \gamma < 2$ ,  $-\sqrt{6} < N < 0$ , and  $M < \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)}$ ; or
- (d) if  $0 < \gamma < \frac{4}{3}$ ,  $0 < N < \frac{4}{3}$ , and  $M < \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)}$ ; or
- (e) if  $\frac{4}{3} < \gamma < 2$ ,  $0 < N < \sqrt{6}$ , and  $M > \frac{2(N^2 - 3\gamma)}{N(3\gamma - 4)}$  the stable manifold is tangent to the  $x_1$ - $x_2$  plane;

(f) By interchanging  $>$  and  $<$  in the inequalities for  $M$  in the cases (b)-(e) we find that stable manifold is 1-dimensional and is tangent to the singular point in the direction of the  $x_1$ -axis (accordingly to that, the unstable subspace is spanned by  $x_2$ -axis).

5. The singular points  $P_{5,6}$  with coordinates  $y = \frac{\sqrt{6}\gamma}{M(3\gamma-4)-2N}$ ,  $z = \mp \frac{\sqrt{4N(2M+N)-6(MN+2)\gamma}}{2N+M(4-3\gamma)}$  (respectively) exists if the following conditions are simultaneously satisfied:  $4N(2M+N) - 6(MN+2)\gamma \geq 0$ ,  $\mp(2N+M(4-3\gamma)) > 0$  and  $\frac{4N^2+M(8-6\gamma)N+6(\gamma-2)\gamma}{(2N+M(4-3\gamma))^2} \leq 1$  (i.e., the singular points are real-valued, and they are inside the cylinder  $\bar{\Sigma}_\epsilon$ ).

The associated eigenvalues are  $\lambda_{5,6}^\pm = \frac{\alpha}{\beta} \pm \frac{\sqrt{8(\beta^2+27\gamma^2)\alpha^2-2\beta(\gamma-4)(\beta^2-216\gamma^2)\alpha-(\gamma-2)(\beta^2-216\gamma^2)^2}}{6\sqrt{6}\beta\gamma}$  and  $\lambda_{5,6} = 0$ , where  $\alpha = 3(N(\gamma-2) + M(3\gamma-4))$  and  $\beta = 2(2N - M(3\gamma-4))$ . Assuming that the conditions for existence are satisfied, we can analyze the stability of the singular points by means of the Center manifold theorem. We find that the non null eigenvalues can not be either complex conjugated with positive real parts or real-valued with different sign, then, the unstable subspace of  $P_{5,6}$  is the empty set. Then, the stable subspace is 2-dimensional (provided  $\lambda_{5,6}$  is the only null eigenvalue). When the orbits are restricted to this invariant set, the point  $P_{5,6}$  acts as an stable spiral (if the eigenvalues are complex conjugated) or as a node (if the eigenvalues are negative reals). The conditions on the parameters for those cases are very complicated to displayed them here. If  $M = \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$  the points  $P_{5,6}$  reduces to  $P_4$  but in this case, the center manifold is 2-dimensional and tangent at the singular point to the  $\varphi$ - $x_2$  plane.

### 3.4.5 The flow in the invariant set $\rho_r = 0$ near $\phi = -\infty$

With the purpose of complementing the global analysis of the system (3.23)-(3.26) defined in  $\Sigma$  it is necessary investigate its behavior near  $\phi = -\infty$ . It is an easy task since the system (3.23)-(3.26) is invariant under the transformation of coordinates

$$(\phi, x_1) \rightarrow (-\phi, -x_1), \quad V \rightarrow U, \quad \chi \rightarrow \Xi, \quad (3.82)$$

where  $U(\phi) = V(-\phi)$  and  $\Xi(\phi) = \chi(-\phi)$ . Hence, for a particular potential  $V$ , and a particular coupling function  $\chi$ , the behavior of the solutions of the equations (3.23)-(3.26) around  $\phi = -\infty$  is equivalent (except for the sign of  $\phi$ ) to the behavior of the system near  $\phi = \infty$  with potential and coupling functions  $U$  and  $\Xi$ , respectively.

If  $U$  and  $\Xi$  are of class  $\mathcal{E}_+^2$ , the preceding analysis in  $\bar{\Sigma}_\epsilon$  can be applied (with and adequate choice of  $\epsilon$ ).

In the following we will denote  $\mathcal{E}^k$  to the set of class  $C^k$  functions well behaved in both  $+\infty$  and  $-\infty$ . We will use Latin uppercase letters with subscripts  $+\infty$  and  $-\infty$ , respectively to indicate the exponential order of  $\mathcal{E}^k$  functions in  $+\infty$  and in  $-\infty$ .

### 3.4.6 The topological structure of the invariant set $\rho_r = 0$ at the past attractor

Let be  $x \in \mathbb{R}_+$  fixed and let be defined the set

$$\Sigma(x) := \{(\phi, x_1, x_2, x_3) \in \Sigma \mid x_3 < x\}. \quad (3.83)$$

Since  $x_3$  is a monotonic increasing function of  $\tau$  follows that  $\Sigma(x)$  is an invariant set and coincides with the union of its past orbits. Hence, in order to investigate the topological properties of the past attractor it is suffice to investigate the topological properties of  $\Sigma(x)$ .

Let  $(V, h)$  a local chart for  $\Sigma$ , and let be  $h|_{V \cap \Sigma(x)}$  the restriction of  $h$  to  $\Sigma(x)$ . Since  $\Sigma(x)$  is an open set with respect to the induced topology, then  $(V \cap \Sigma(x), h|_{V \cap \Sigma(x)})$  is a local chart for  $\Sigma(x)$ . From this fact follows that

**Remark 3**  $\Sigma(x)$  is a topological manifold with boundary

$$\partial \Sigma(x) := \{\mathbf{p} \in \Sigma \mid \phi \in \mathbb{R}, x_2 = 0, 0 \leq x_3 < x\} \cup \{\phi \in \mathbb{R}, x_2 \geq 0, x_3 = 0\},$$

for all  $x \in \mathbb{R}_+$ .

In order to describing the global behavior of the system (3.23)-(3.26) towards the past, it is required to make the immersion of  $\Sigma(x)$  in a compact differentiable 4-dimensional manifold  $\Omega(x)$ , such that the vector field defined by (3.23)-(3.26) can be smoothly extended over  $\Omega(x)$ . To proceed forward, we define the covering of  $\Sigma(x)$  by open sets with respect to the induced topology as follows. Let be  $\epsilon > 0$ , a real number, and let be defined the sets

$$\Sigma(x, \epsilon)^- = \{\mathbf{p} \in \Sigma(x) \mid \phi < -\epsilon^{-1}\}, \quad (3.84)$$

$$\Sigma(x, \epsilon) = \{\mathbf{p} \in \Sigma(x) \mid -1 - \epsilon^{-1} < \phi < 1 + \epsilon^{-1}\}, \quad (3.85)$$

$$\Sigma(x, \epsilon)^+ = \{\mathbf{p} \in \Sigma(x) \mid \phi > \epsilon^{-1}\}. \quad (3.86)$$

From the inequality  $-1 - \epsilon^{-1} < -\epsilon^{-1} < 1 + \epsilon^{-1}$  follows that (3.84)-(3.86) is a covering of  $\Sigma(x)$ . By construction they are open sets with respect to the induced topology.

Let be defined the sets:

$$\begin{aligned} \Omega(x, \epsilon) = \{(\varphi, x_1, x_2, x_3) \in \mathbb{R}^4 \mid & -1 - \epsilon^{-1} < \varphi < 1 + \epsilon^{-1}, \\ & -1 < x_1 < 1, 0 < x_2 < 1, 0 < x_3 < x\}. \end{aligned} \quad (3.87)$$

This set contains a submanifold of dimension 3 which is homeomorphic under the identity map to  $\Sigma(x, \epsilon)$ .

$$\begin{aligned}\Omega(x, \epsilon)^+ &= \{ \mathbf{p} = (\varphi, x_1, x_2, x_3) \in \mathbb{R}^4 | 0 < \varphi < f(\epsilon^{-1}), \\ &\quad -1 < x_1 < 1, 0 < x_2 < 1, 0 < x_3 < x \},\end{aligned}\quad (3.88)$$

(where  $f$  is the function defined in definition (27)). This set contains a submanifold of dimension 3 which is homeomorphic under the map

$$(\varphi, x_1, x_2, x_3) \xrightarrow{\phi=f^{-1}(\varphi)} \left( \phi, x_1, x_2, \sqrt{\frac{3(1-x_1^2-x_2^2)}{V(\phi)}} \right)$$

to  $\Sigma(x, \epsilon)^+$ . Finally, it is constructed a set  $\Omega(x, \epsilon)^-$  in an analogous way as proceeded to construct  $\Omega(x, \epsilon)^+$  using the successive coordinate transformations (3.82) and (3.77) with the identifications  $U(\phi) = V(-\phi)$  and  $\Xi(\phi) = \chi(-\phi)$ . The interior of  $\Omega(x)$  it is defined by

$$\Omega(x, \epsilon)^- \cup \Omega(x, \epsilon) \cup \Omega(x, \epsilon)^+.$$

The construction is completed by attaching a boundary, denoted by  $\partial\Omega(x)$ , which is defined taking the union of  $x_3 = 0$ ,  $x_3 = x$ ,  $\varphi = 0$  and the circumference  $x_1^2 + x_2^2 = 1$  to each local chart.

By construction,  $\Omega(x)$  is compact and it is embedded in  $\mathbb{R}^4$ .

Thus, the vector field defined by (3.23)-(3.26) can be smoothly extended over the boundary of  $\Omega(x)$  such that  $\Omega(x)$  is the union of its past orbits. It is important to note that  $\Omega(x)$  approaches the non-physical boundary  $\partial\Omega$ , along the intersection of the plane  $x_3 = 0$  with the plane  $\varphi = 0$  and the circumference  $x_1^2 + x_2^2 = 1$ .

### 3.4.6.1 The initial space-time singularity.

In this section we will study the initial space-time (Big-Bang) singularity. The singular points  $P_{1,2}$  can represent such a singularity. They live at the same phase space for the values of  $M$ ,  $N$  and  $\gamma$  in the intervals  $-\sqrt{6} < N < \sqrt{6}$ ,  $-\frac{\sqrt{6}(\gamma-2)}{3\gamma-4} < M < \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$  and  $0 < \gamma < \frac{4}{3}$  (in this case, they have an unstable 2-dimensional manifold and a center 1-dimensional manifold). It is easy to show that the Hubble parameter and the matter energy density of the associated cosmological solutions diverge towards the past. The scalar field diverges too, and it is equal to  $+\infty$  and to  $-\infty$  for  $P_1$  and  $P_2$  respectively. However, even in this case, the possible past attractor corresponds to  $P_1$  since  $\bar{f}' < 0$  whereas for  $x_1 > 0$  the orbits enter the phase space and  $P_2$  acts as a saddle. The singular point  $P_2$  can act as a past attractor only in a set of initial conditions of measure zero (when  $\varphi = 0$ ).

**Analysis near  $P_1$ .** From the analysis in section 3.4.4.1, it seems reasonable to think that the initial space-time singularity can be associated to the singular point  $P_1$ . Its unstable manifold is 2-dimensional provided  $N < \sqrt{6}$ .

The asymptotic behavior of neighboring solutions to  $P_1$  can be approximated, for  $\tau$  negative large enough, as

$$x_1(\tau) = -1 + O(e^{\lambda_{1,1}\tau}), \quad x_2(\tau) = O(e^{\lambda_{1,2}\tau}). \quad (3.89)$$

By substitution of (3.89) in (3.23), and integrating the resulting equation, we obtain

$$\phi(\tau) = \sqrt{\frac{2}{3}} \left( -\tau + \tilde{\phi} \right) + O(e^{\lambda_{1,1}\tau}). \quad (3.90)$$

Then, by expanding around  $\tau = -\infty$  up to first order, we get

$$\begin{aligned} \varphi &= f \left( \sqrt{\frac{2}{3}} \left( -\tau + \tilde{\phi} \right) + O\left(\frac{1}{\tau}\right)^2 \right) + O(e^{\lambda_{1,1}\tau}) \\ &= f \left( \sqrt{\frac{2}{3}} \left( -\tau + \tilde{\phi} \right) \right) + O(e^{\lambda_{1,1}\tau}) + h, \end{aligned}$$

where  $h$  denotes higher order terms to be discarded.

Then we have a first order solution to (3.78-3.80). Also, by substitution of (3.89) in (3.26) and solving the resulting differential equation with initial condition  $x(0) = x_0$  we get the first order solution

$$x = x_0 e^\tau. \quad (3.91)$$

Then, we have  $t - t_i = \frac{1}{3} \int x(\tau) d\tau = 1/3 x_0 e^\tau$ . For simplicity let us set  $t_i = 0$ .

Neglecting the error terms, we have the following expressions

$$\begin{aligned} H &= x^{-1} = (x_0 e^\tau)^{-1} = \frac{1}{3t}, \quad \phi = \sqrt{\frac{2}{3}} \left( -\tau + \tilde{\phi} \right) = -\sqrt{\frac{2}{3}} \ln \frac{t}{c}, \\ \dot{\phi} &= -\sqrt{\frac{2}{3}} t^{-1}, \quad \rho = 0, \end{aligned} \quad (3.92)$$

where  $c = 1/3x_0 e^{\tilde{\phi}}$ . This asymptotic solution corresponds to the exact solution of the cosmological equations when  $V$  vanishes identically and  $\chi$  is a constant (the minimal coupling case). Hence, there exists a generic class of massless minimally coupled scalar field cosmologies in a vicinity of the initial space-time singularity.

The above idea can be stated, more precisely, as the

**Theorem 24 (Local singularity theorem)** *Let be  $V \in \mathcal{E}_+^2$  with exponential order  $N$  such that  $N < \sqrt{6}$  and let be  $\chi \in \mathcal{E}_+^2$  with exponential order  $M$  such that*

$$I. \quad 0 < \gamma < \frac{4}{3} \text{ y } M > (2N - \sqrt{6}\gamma) / (3\gamma - 4) \text{ or}$$

$$2. \frac{4}{3} < \gamma < 2 \text{ y } M < (2N - \sqrt{6}\gamma) / (3\gamma - 4)$$

Then, there exists a neighborhood  $\mathcal{N}(P_1)$  of  $P_1$  such that for all  $p \in \mathcal{N}(P_1)$ , the orbit  $\psi_p$  is past asymptotic to  $P_1$  and the associated cosmological solution is given by:

$$H = \frac{1}{3t} + O(\varepsilon_V(t)), \quad (3.93)$$

$$\phi = -\sqrt{\frac{2}{3}} \ln \frac{t}{c} + O(t\varepsilon_V(t)), \quad (3.94)$$

$$\dot{\phi} = -\sqrt{\frac{2}{3}} t^{-1} + O(\varepsilon_V(t)), \quad (3.95)$$

$$\rho = \frac{b_0^2}{3} t^{-\gamma} \chi \left( -\sqrt{\frac{2}{3}} \ln \frac{t}{c} \right)^{\frac{3\gamma}{2}-2} (1 + O(t\varepsilon_V(t))), \quad (3.96)$$

where  $\varepsilon_V(t) = tV \left( -\sqrt{\frac{2}{3}} \ln \frac{t}{c} \right)$ .

**Comments.** Since  $V \in \mathcal{E}_+^2$  has exponential order  $N$ , then, applying theorem 23, we have

$$\lim_{t \rightarrow 0} t^\alpha V \left( -\sqrt{\frac{2}{3}} \ln \frac{t}{c} \right) = \lim_{\phi \rightarrow \infty} e^{-\sqrt{\frac{3}{2}}\alpha\phi} V(\phi) = 0, \quad \forall \alpha > \sqrt{\frac{2}{3}}N.$$

Thus, for  $N < \sqrt{6}$ , the error terms  $O(\varepsilon_V(t))$  and  $O(t\varepsilon_V(t))$  in (3.93)-(3.96) are dominated by the first order terms. If  $N < \sqrt{\frac{3}{2}}$ , both terms tends to zero as  $t \rightarrow 0^+$ .

Since  $\chi \in \mathcal{E}_+^2$  has exponential order  $M$ , then:

1. The term  $t^{-\gamma} \chi \left( -\sqrt{\frac{2}{3}} \ln \frac{t}{c} \right)^{\frac{3\gamma}{2}-2} O(t\varepsilon_V(t)) = O(t^{2-\gamma+\frac{M(4-3\gamma)}{\sqrt{6}}-\sqrt{\frac{2}{3}}N})$  tends to zero as  $t \rightarrow 0^+$ , in the cases

- (a)  $0 < \gamma < \frac{4}{3}$ ,  $N \leq \sqrt{\frac{3}{2}}$ ,  $M > \frac{(2N-\sqrt{6}\gamma)}{3\gamma-4}$  or
- (b)  $0 < \gamma < \frac{4}{3}$ ,  $\sqrt{\frac{3}{2}} < N \leq \sqrt{6}$ ,  $M > -\frac{(2N-\sqrt{6}(2-\gamma))}{3\gamma-4}$  or
- (c)  $\frac{4}{3} < \gamma < 2$ ,  $N \leq \sqrt{\frac{3}{2}}$ ,  $M < \frac{(2N-\sqrt{6}\gamma)}{3\gamma-4}$  or
- (d)  $\frac{4}{3} < \gamma < 2$ ,  $\sqrt{\frac{3}{2}} < N \leq \sqrt{6}$ ,  $M < -\frac{(2N-\sqrt{6}(2-\gamma))}{3\gamma-4}$ .

2. The term  $t^{-\gamma} \chi \left( -\sqrt{\frac{2}{3}} \ln \frac{t}{c} \right)^{\frac{3\gamma}{2}-2}$  tends to zero as  $t \rightarrow 0^+$ , in the cases

- (a)  $0 < \gamma < \frac{4}{3}$ ,  $M > -\frac{\sqrt{6}\gamma}{3\gamma-4}$  or
- (b)  $\frac{4}{3} < \gamma < 2$ ,  $M < -\frac{\sqrt{6}\gamma}{3\gamma-4}$ .



First, us prove the

**Lemma 2** *Let be  $V \in \mathcal{E}_+^k$  a function of exponential order  $N$ . Let  $n > \sqrt{\frac{2}{3}}N$  and  $\lambda > 0$ . Let  $\varphi = f(\phi)$  the coordinate transformation referred in the definition (27). Then, if  $\tau \rightarrow -\infty$ , are valid the estimates*

$$\int_{-\infty}^{\tau} \overline{V}(\varphi) e^{n\tau} d\tau = \frac{3}{3n - \sqrt{6}N} \overline{V}(\varphi) e^{n\tau} + h, \quad (3.97)$$

and

$$\int_{-\infty}^{\tau} e^{\lambda\tau} (\overline{W}_V(\varphi) + N) d\tau = \frac{N}{\lambda} e^{\lambda\tau} + h. \quad (3.98)$$

where  $h$  denotes terms of higher order to be discarded.

**Proof.** We proceed similarly as in the proof of theorem 4 in [403].

1. Let us define the function  $T(\phi) = V(\phi)e^{-N\phi}$ .<sup>9</sup> Since  $V$  is of exponential order  $N$  follows from  $\phi = f^{-1}(\varphi(\tau)) \approx -\sqrt{\frac{2}{3}}(\tau - \tilde{\phi})$  that

$$\int \overline{V}(\varphi) e^{n\tau} d\tau = \int \overline{T}(\varphi) e^{(-\sqrt{\frac{2}{3}}N+n)\tau} d\tau + h$$

where  $h$  denotes terms of higher order to be discarded. By an explicit computation we have that

$$d \ln \overline{T}(\varphi) = \frac{1}{f'(f^{-1}(\varphi))} \left( \frac{V'(f^{-1}(\varphi))}{V(f^{-1}(\varphi))} - N \right) d\varphi = \frac{\overline{W}_V(\varphi)}{\overline{f}'(\varphi)} d\varphi. \quad (3.99)$$

Using (3.78) and (3.99) we obtain

$$d\overline{T}(\varphi) = \sqrt{\frac{2}{3}} x_1 \overline{T}(\varphi) \overline{W}_V(\varphi) d\tau. \quad (3.100)$$

Integrating by parts gives

$$I(\tau) = \int_{-\infty}^{\tau} \overline{T}(\varphi) e^{(-\sqrt{\frac{2}{3}}N+n)\tau} d\tau = I_1(\tau) + I_2(\tau) \quad (3.101)$$

where

$$\begin{aligned} I_1(\tau) &= \frac{3}{3n - \sqrt{6}N} \overline{T}(\varphi) e^{(-\sqrt{\frac{2}{3}}N+n)\tau} \Big|_{-\infty}^{\tau} \\ &= \frac{3}{3n - \sqrt{6}N} \overline{V}(\varphi) e^{n\tau} + h, \end{aligned}$$

---

<sup>9</sup>If  $V$  is of exponential order  $N$ , then  $T$  is of exponential order zero since  $W_T(\phi) = T'(\phi)/T(\phi) = W_V(\phi) \rightarrow 0$  as  $\phi \rightarrow +\infty$ . Hence, for all  $\lambda > 0$ ,  $\lim_{\tau \rightarrow +\infty} e^{-\lambda\phi} T(\phi) = 0$ .

and

$$\begin{aligned} I_2(\tau) &= -\frac{3}{3n - \sqrt{6}N} \int_{-\infty}^{\tau} e^{(-\sqrt{\frac{2}{3}}N+n)\tau} d\overline{T}(\varphi) \\ &= -\frac{2}{\sqrt{6}n - 2N} \int_{-\infty}^{\tau} e^{(-\sqrt{\frac{2}{3}}N+n)\tau} x_1 \overline{T}(\varphi) \overline{W}_V(\varphi) d\tau. \end{aligned}$$

Consider the second term  $I_2(\tau)$ . Let  $\delta(\tau) = \sup_{\tau' < \tau} \frac{2}{\sqrt{6}n - 2N} |\overline{W}_V(\varphi(\tau'))|$ , then, recalling that  $x_1^2 < 1$ ;

$$\begin{aligned} |I_2(\tau)| &= \frac{2}{\sqrt{6}n - 2N} \left| \int_{-\infty}^{\tau} e^{(-\sqrt{\frac{2}{3}}N+n)\tau} x_1 \overline{T}(\varphi) \overline{W}_V(\varphi) d\tau \right| \\ &< \delta(\tau) |I(\tau)| \\ &\leq \delta(\tau) (|I_1(\tau)| + |I_2(\tau)|) \\ &< \frac{\delta(\tau)}{1 - \delta(\tau)} |I_1(\tau)|. \end{aligned}$$

for  $\tau$  sufficiently small. Letting  $\tau$  go to  $-\infty$  demonstrates (3.97), as required.

2. Let

$$I(\tau) = \int_{-\infty}^{\tau} e^{\lambda\tau} \frac{V'(f^{-1}(\varphi))}{V(f^{-1}(\varphi))} d\tau.$$

Integrating by parts we have that

$$\begin{aligned} I(\tau) &= \frac{1}{\lambda} e^{\lambda\tau} \frac{V'(f^{-1}(\varphi))}{V(f^{-1}(\varphi))} \Big|_{-\infty}^{\tau} - \frac{1}{\lambda} \int_{-\infty}^{\tau} e^{\lambda\tau} d \left[ \frac{V'(f^{-1}(\varphi))}{V(f^{-1}(\varphi))} \right] \\ &= \frac{1}{\lambda} e^{\lambda\tau} (\overline{W}_V(\varphi) + N) \Big|_{-\infty}^{\tau} - \frac{1}{\lambda} \int_{-\infty}^{\tau} e^{\lambda\tau} d \left[ \frac{V'(f^{-1}(\varphi))}{V(f^{-1}(\varphi))} \right] \\ &= I_1(\tau) + I_2(\tau), \end{aligned}$$

where  $I_1(\tau) = \frac{1}{\lambda} e^{\lambda\tau} (\overline{W}_V(\varphi) + N) \Big|_{-\infty}^{\tau} = \frac{1}{\lambda} e^{\lambda\tau} (\overline{W}_V(\varphi) + N) = \frac{N}{\lambda} e^{\lambda\tau} + h$ , since  $\overline{W}_V(\varphi)$  goes to zero, as  $\tau \rightarrow -\infty$ ; and

$$\begin{aligned} I_2(\tau) &= -\frac{1}{\lambda} \int_{-\infty}^{\tau} e^{\lambda\tau} d \left[ \frac{V'(f^{-1}(\varphi))}{V(f^{-1}(\varphi))} \right] \\ &= -\frac{1}{\lambda} \int_{-\infty}^{\tau} e^{\lambda\tau} g(\varphi) \left( \frac{d(\ln(g(\varphi)))}{d\varphi} \frac{d\varphi}{d\tau} \right) d\tau \\ &= -\frac{1}{\lambda} \sqrt{\frac{2}{3}} \int_{-\infty}^{\tau} e^{\lambda\tau} g(\varphi) \left( \frac{d(\ln(g(\varphi)))}{d\varphi} \overline{f}'(\varphi) x_1 \right) d\tau. \end{aligned}$$

where

$$g(\varphi) = \overline{W}_V(\varphi) + N.$$

Let

$$\delta_1(\tau) = \sup_{\tau' < \tau} \frac{1}{\lambda} \sqrt{\frac{2}{3}} \left| \bar{f}'(\varphi) \frac{d}{d\varphi} (\ln (\overline{W_V}(\varphi(\tau')) + N)) \right|,$$

then, recalling that  $x_1^2 < 1$ ;

$$\begin{aligned} |I_2(\tau)| &< \delta(\tau) |I(\tau)| \\ &\leq \delta(\tau) (|I_1(\tau)| + |I_2(\tau)|) \\ &< \frac{\delta(\tau)}{1 - \delta(\tau)} |I_1(\tau)|. \end{aligned}$$

for  $\tau$  sufficiently small. Letting  $\tau$  go to  $-\infty$  demonstrates (3.98), as required.

■

**Proof of theorem 24** Let us consider the system given by the differential equations (3.78)-(3.80) and (3.26). Using the restriction (3.81) and making the coordinate transformation  $x_1 \rightarrow -1 + u$  the system (3.78)-(3.80), (3.26) can be expressed in integral form as

$$\varphi(\tau) = -\sqrt{\frac{2}{3}} \int_{-\infty}^{\tau} \bar{f}'(\varphi(s)) (1 - u(s)) ds \quad (3.102)$$

$$u(\tau) = u_0 e^{\lambda_{1,1}\tau} + \int_{-\infty}^{\tau} e^{\lambda_{1,1}(\tau-s)} h_1(\varphi(s), u(s), x_2(s), x_3(s)) ds, \quad (3.103)$$

$$x_2(\tau) = x_{20} e^{\lambda_{1,2}\tau} + \int_{-\infty}^{\tau} e^{\lambda_{1,2}(\tau-s)} h_2(\varphi(s), u(s), x_2(s), x_3(s)) ds, \quad (3.104)$$

$$x_3(\tau) = x_{30} e^{\tau} + \int_{-\infty}^{\tau} e^{(\tau-s)} h_3(\varphi(s), u(s), x_2(s), x_3(s)) ds, \quad (3.105)$$

for  $\tau \rightarrow -\infty$ . Where

$$\begin{aligned} h_1(\varphi, u, x_2, x_3) &= u^3 - 3u^2 + \frac{\gamma x_2^2}{2} (u - 1) - \sqrt{\frac{2}{3}} u \overline{W_V}(\varphi) + \\ &+ \left( \frac{u^2}{\sqrt{6}} + \frac{x_2^2}{\sqrt{6}} \right) (N + \overline{W_V}(\varphi)) + \frac{x_2^2(4 - 3\gamma)(M + \overline{W_X}(\varphi))}{2\sqrt{6}}, \end{aligned} \quad (3.106)$$

$$\begin{aligned} h_2(\varphi, u, x_2, x_3) &= x_2 u^2 - 2x_2 u + \frac{1}{2} \gamma x_2^3 - \frac{x_2(3\gamma - 4)\overline{W_X}(\varphi)}{2\sqrt{6}} + \\ &+ \frac{u x_2(3\gamma - 4)(M + \overline{W_X}(\varphi))}{2\sqrt{6}}, \end{aligned} \quad (3.107)$$

$$h_3(\varphi, u, x_2, x_3) = -\frac{1}{3} x_3^3 \overline{V}(\varphi) - \frac{1}{2} (2 - \gamma) x_2^2 x_3. \quad (3.108)$$

To derive (3.102)-(3.105) we have used the fact that

$$\lim_{\tau \rightarrow -\infty} (\varphi(\tau), e^{-\lambda_{1,1}\tau} u(\tau), e^{-\lambda_{1,2}\tau} x_2(\tau), e^{-\tau} x_3(\tau)) = (0, x_{10}, x_{20}, x_{30}),$$

where  $u_0 > 0, x_{20} > 0, x_{30} \neq 0$  are sufficiently small real constants. This fact can be proved using the first order approximations, as  $\tau \rightarrow -\infty$ :  $f^{-1}(\varphi(\tau)) \approx -\sqrt{\frac{2}{3}}(\tau - \tilde{\phi})$ ,  $u(\tau) = u_0 e^{\lambda_{1,1}\tau}$ ,  $x_2(\tau) = x_{20} e^{\lambda_{1,2}\tau}$ ,  $x_3(\tau) = x_{30} e^\tau$ .

To obtain the required expansions, we substitute in the equations (3.102)-(3.105) the above first order approximations; then, will be suffice to estimate integrals of types

$$\int_{-\infty}^{\tau} \overline{V}(\varphi(s)) e^{n\tau} d\tau, \int_{-\infty}^{\tau} \overline{W}_V(\varphi(s)) ds, \int_{-\infty}^{\tau} e^{\lambda s} (N + \overline{W}_V(\varphi(s))) ds, \quad (3.109)$$

as  $\tau \rightarrow -\infty$  where  $n = 2$  and  $\lambda = \{\lambda_{1,1}, 2\lambda_{1,2} - \lambda_{1,1}\}$  where we are assuming that  $V$  is a generic WBI function of exponential order  $N$  and  $\lambda > 0$ .

Since  $\overline{W}_V(\varphi)$  goes to zero, as  $\tau \rightarrow -\infty$ ; the second integral in (3.109) is an infinitesimal of higher order as  $\tau \rightarrow -\infty$ . In fact,  $\int_{-\infty}^{\tau} \overline{W}_V(\varphi) d\tau = \int_{-\infty}^{\tau} \overline{W}_T(\varphi) d\tau + h = \int_{-\infty}^{\tau} \frac{T'(f^{-1}(\varphi))}{T(f^{-1}(\varphi))} d\tau + h$ . Using the first order solution  $\phi = f^{-1}(\varphi(\tau)) \approx -\sqrt{\frac{2}{3}}(\tau - \tilde{\phi})$ , the variation of constants formulae and the smoothness of  $T(\phi)$  as  $\phi \rightarrow +\infty$  we have that  $\int_{-\infty}^{\tau} \overline{W}_V(\varphi) d\tau = -\sqrt{\frac{2}{3}} \ln \left[ \frac{T(-\sqrt{\frac{2}{3}}(\tau - \tilde{\phi}))}{\lim_{\phi \rightarrow +\infty} T(\phi)} \right] \rightarrow 0$  as  $\tau \rightarrow -\infty$ .

Using the above procedure and the results of lemma 2 we obtain

$$\begin{aligned} x_3(\tau) &= x_{30} e^\tau - \frac{x_{30}^3}{3} e^\tau \int_{-\infty}^{\tau} e^{2s} \overline{V}(\varphi(s)) ds + \\ &\quad - \frac{1}{2} (2 - \gamma) x_{20}^2 x_{30} e^\tau \int_{-\infty}^{\tau} e^{2\lambda_{1,2}s} ds \\ &= x_{30} e^\tau - \frac{x_{30}^3}{3\lambda_{1,1}} e^\tau e^{2\tau} \overline{V}(\varphi(\tau)) - \frac{(2 - \gamma) x_{20}^2 x_{30}}{4\lambda_{1,2}} e^\tau e^{2\lambda_{1,2}\tau} + h. \end{aligned} \quad (3.110)$$

Since  $V$  is of exponential order  $N$  follows (at first order) that  $e^{2\tau} \overline{V}(\varphi) = O(e^{\lambda_{1,1}\tau})$ . Then, from the hypothesis  $2\lambda_{1,2} > \lambda_{1,1}$  follows that the third term in the second line of equation (3.110) is an infinitesimal or higher order than the second one as  $\tau \rightarrow -\infty$ . Thus,

$$x_3(\tau) = x_{30} e^\tau - \frac{(x_{30} e^\tau)^3}{3\lambda_{1,1}} \overline{V}(\varphi(\tau)) + h. \quad (3.111)$$

Integrating again to obtain a second order expression for  $t$  and using the fact that  $t \rightarrow 0^+$  as  $\tau \rightarrow -\infty$ , we get

$$\begin{aligned} 3t &= \int_{-\infty}^{\tau} x_3(s) ds = x_{30} e^\tau - \frac{x_{30}^3}{3\lambda_{1,1}} \int_{-\infty}^{\tau} e^{3s} \overline{V}(\varphi(s)) ds + h \\ &= x_{30} e^\tau - \frac{x_{30}^3}{\lambda_{1,1}(9 - \sqrt{6}N)} e^{3\tau} \overline{V}(\varphi(\tau)) + h \\ &= (x_{30} e^\tau) - \frac{(x_{30} e^\tau)^3}{\lambda_{1,1}(9 - \sqrt{6}N)} \overline{V}(\varphi(\tau)) + h \end{aligned} \quad (3.112)$$

This expression may be inverted, to second order, to give:

$$x_{30}e^\tau = 3t + \frac{27t^3}{\lambda_{1,1}(9 - \sqrt{6}N)}V(\phi(t)) + h, \quad (3.113)$$

Applying the inversion formula to  $x_3$  as given by formula (3.111) follows:

$$x_3(t) = 3t - \frac{27}{(9 - \sqrt{6}N)}t^3V(\phi(t)) + h. \quad (3.114)$$

Hence,

$$H(t) = x_3(t)^{-1} = \frac{1}{3t} + \frac{3V(\phi(t))t}{(9 - \sqrt{6}N)} + h. \quad (3.115)$$

Using the formula (3.113) we can obtain a second order approximation for  $\rho$  as follows.

Integrating equation (3.15) for  $\rho$  we obtain the expression

$$\rho = \rho_0 a^{-3\gamma} \chi^{-3+\frac{3\gamma}{2}},$$

where  $\rho_0$  is an integration constant. Recalling  $\tau = \ln a^3$  follows that

$$\rho = \frac{\rho_0 x_{30}^\gamma \chi(\phi)^{-3+\frac{3\gamma}{2}}}{(x_{30}e^\tau)^\gamma}.$$

Substituting back the formula (3.113) we obtain

$$\rho(t) = \frac{1}{3}b_0^2 t^{-\gamma} \left( \frac{9\gamma V(\phi(t))t^2}{(\sqrt{6}N - 9)\lambda_{1,1}} + 1 \right) \chi(\phi(t))^{\frac{3\gamma}{2}-2}, \quad (3.116)$$

where  $b_0 = 3^{\frac{1-\gamma}{2}} x_{30}^{\gamma/2} \sqrt{\rho_0}$ .

In an analogous way as we have obtained a second order estimate for  $x_3(\tau)$  we proceed for  $u(\tau)$ .

$$\begin{aligned} u(\tau) = & \frac{e^{3\lambda_{1,1}\tau} u_0^3}{2\lambda_{1,1}} + \frac{e^{2\lambda_{1,1}\tau} (\sqrt{6}N - 18) u_0^2}{6\lambda_{1,1}} + e^{\lambda_{1,1}\tau} u_0 + \\ & + x_{20}^2 \left( \frac{e^{(\lambda_{1,1}+2\lambda_{1,2})\tau} u_0^\gamma}{4\lambda_{1,2}} + \frac{1}{2} e^{2\lambda_{1,2}\tau} \right) \end{aligned} \quad (3.117)$$

We have the following three cases:

1.

$$\lambda_{1,2} < \lambda_{1,1} < 2\lambda_{1,2}$$

implies

$$\lambda_{1,1} < 2\lambda_{1,2} < 2\lambda_{1,1} < \lambda_{1,1} + 2\lambda_{1,2} < 3\lambda_{1,1};$$

thus

$$u(\tau) = e^{\lambda_{1,1}\tau} u_0 + \frac{x_{20}^2}{2} e^{2\lambda_{1,2}\tau} + h. \quad (3.118)$$

Then, from the hypothesis  $2\lambda_{1,2} > \lambda_{1,1}$  follows that the second term in equation (3.118) is an infinitesimal or higher order than the first one as  $\tau \rightarrow -\infty$ . Thus

$$u(\tau) = e^{\lambda_{1,1}\tau} u_0 + h. \quad (3.119)$$

2.

$$\frac{2\lambda_{1,2}}{3} < \lambda_{1,1} \leq \lambda_{1,2}$$

implies

$$\lambda_{1,1} < 2\lambda_{1,1} \leq 2\lambda_{1,2} < 3\lambda_{1,1} < \lambda_{1,1} + 2\lambda_{1,2};$$

thus

$$u(\tau) = e^{\lambda_{1,1}\tau} u_0 + \frac{e^{2\lambda_{1,1}\tau} (\sqrt{6}N - 18) u_0^2}{6\lambda_{1,1}} + \frac{x_{20}^2}{2} e^{2\lambda_{1,2}\tau} + h \quad (3.120)$$

Then, from the hypothesis  $2\lambda_{1,2} > \lambda_{1,1}$  follows that the third term in equation (3.120) is an infinitesimal or higher order than the second one as  $\tau \rightarrow -\infty$ . Thus

$$u(\tau) = e^{\lambda_{1,1}\tau} u_0 + \frac{e^{2\lambda_{1,1}\tau} (\sqrt{6}N - 18) u_0^2}{6\lambda_{1,1}} + h \quad (3.121)$$

3.

$$0 < \lambda_{1,1} \leq \frac{2\lambda_{1,2}}{3}$$

implies

$$\lambda_{1,1} < 2\lambda_{1,1} < 3\lambda_{1,1} \leq 2\lambda_{1,2} < \lambda_{1,1} + 2\lambda_{1,2};$$

thus

$$u(\tau) = e^{\lambda_{1,1}\tau} u_0 + \frac{e^{2\lambda_{1,1}\tau} (\sqrt{6}N - 18) u_0^2}{6\lambda_{1,1}} + \frac{e^{3\lambda_{1,1}\tau} u_0^3}{2\lambda_{1,1}} + \frac{x_{20}^2}{2} e^{2\lambda_{1,2}\tau} + h \quad (3.122)$$

Then, from the hypothesis  $2\lambda_{1,2} > \lambda_{1,1}$  follows that the fourth term in equation (3.122) is an infinitesimal or higher order than the third one as  $\tau \rightarrow -\infty$ . Thus

$$u(\tau) = e^{\lambda_{1,1}\tau} u_0 + \frac{e^{2\lambda_{1,1}\tau} (\sqrt{6}N - 18) u_0^2}{6\lambda_{1,1}} + \frac{e^{3\lambda_{1,1}\tau} u_0^3}{2\lambda_{1,1}} + h \quad (3.123)$$

In the previous three cases

$$u(\tau) = P(e^{\lambda_{1,1}\tau}u_0) + h$$

with  $P(x) = x + Ax^2 + Bx^3$ , where  $A = B = 0$  in case 1;  $A = \frac{\sqrt{6}N-18}{6\lambda_{1,1}}$  and  $B = 0$  in case 2; and in the case 3,  $A = \frac{\sqrt{6}N-18}{6\lambda_{1,1}}$ ,  $B = \frac{1}{2\lambda_{1,1}}$ . Then,

$$u(t) = P(a_0 t^{\lambda_{1,1}}) - \frac{9a_0 t^{\lambda_{1,1}+2} (2Aa_0 t^{\lambda_{1,1}} + 3a_0^2 B t^{2\lambda_{1,1}} + 1) V(\phi(t))}{\sqrt{6}n - 9}, \quad (3.124)$$

where  $a_0 = 3^{\lambda_{1,1}} u_0 \left( \frac{1}{x_{30}} \right)^{\lambda_{1,1}}$ .

Substituting the expansions (3.124) and (3.114) in the expression

$$\dot{\phi} = \frac{\sqrt{6}(u - 1)}{x_3}$$

we obtain

$$\begin{aligned} \dot{\phi} = & -\frac{\sqrt{\frac{2}{3}}}{t} + \frac{3\sqrt{6}V(\phi(t))t}{\sqrt{6}N - 9} + \left( \frac{\sqrt{\frac{2}{3}}a_0}{t} - \frac{6\sqrt{6}a_0 t V(\phi(t))}{\sqrt{6}N - 9} \right) t^{\lambda_{1,1}} + \\ & + \left( \frac{\sqrt{\frac{2}{3}}Aa_0^2}{t} - \frac{9\sqrt{6}Aa_0^2 t V(\phi(t))}{\sqrt{6}N - 9} \right) t^{2\lambda_{1,1}} + \\ & + \left( \frac{\sqrt{\frac{2}{3}}a_0^3 B}{t} - \frac{12\sqrt{6}a_0^3 B t V(\phi(t))}{\sqrt{6}N - 9} \right) t^{3\lambda_{1,1}} + h \end{aligned} \quad (3.125)$$

Finally, from

$$\phi'(\tau) \equiv \sqrt{\frac{2}{3}}(u(\tau) - 1),$$

and (3.123) follows, by integration,

$$\begin{aligned} \phi(\tau) = & -\sqrt{\frac{2}{3}}(\tau - \tilde{\phi}) + \sqrt{\frac{2}{3}} \int_{-\infty}^{\tau} u(\tau) d\tau \\ = & -\sqrt{\frac{2}{3}}(\tau - \tilde{\phi}) + \frac{e^{3\lambda_{1,1}\tau} u_0^3}{3\sqrt{6}\lambda_{1,1}^2} + \frac{e^{2\lambda_{1,1}\tau} (N - 3\sqrt{6}) u_0^2}{6\lambda_{1,1}^2} + \\ & + \frac{\sqrt{\frac{2}{3}} e^{\lambda_{1,1}\tau} u_0}{\lambda_{1,1}}. \end{aligned} \quad (3.126)$$

Using the inversion formula (3.113) we obtain

$$\begin{aligned}
\phi(t) = & -\sqrt{\frac{2}{3}} \ln \left( \frac{t}{c} \right) + \left( \frac{\sqrt{\frac{2}{3}} a_0}{\lambda_{1,1}} - \frac{3\sqrt{6} a_0 t^2 V(\phi(t))}{(\sqrt{6}N - 9) \lambda_{1,1}} \right) t^{\lambda_{1,1}} + \\
& + \left( \frac{a_0^2 (N - 3\sqrt{6})}{6\lambda_{1,1}^2} - \frac{3a_0^2 (N - 3\sqrt{6}) t^2 V(\phi(t))}{(\sqrt{6}N - 9) \lambda_{1,1}^2} \right) t^{2\lambda_{1,1}} + \\
& + \left( \frac{a_0^3}{3\sqrt{6}\lambda_{1,1}^3} - \frac{3\sqrt{\frac{3}{2}} a_0^3 t^2 V(\phi(t))}{(\sqrt{6}N - 9) \lambda_{1,1}^3} \right) t^{3\lambda_{1,1}}
\end{aligned} \tag{3.127}$$

where  $c = 1/3x_0 e^{\tilde{\phi}}$ .

Now, Taylor-expanding  $V$  and  $\chi$  around  $\phi^* = -\sqrt{\frac{2}{3}} \ln \left( \frac{t}{c} \right)$  and substituting the results in the left hand side of equations (3.115), (3.127), (3.125), and (3.116) completes the proof. ■

### 3.4.6.2 A global singularity theorem

Finally, we will state (without a rigorous proof) a global singularity theorem which is in some way an extension of Theorem 6 in [403] (page 3501). It is not totally an extension of this theorem, since in our framework it is very difficult to prove that the correspondence with the massless minimally coupled scalar field cosmologies is one-to-one.

The theorem states the following:

**Theorem 25 (Global singularity theorem)** *Let be  $V \in \mathcal{E}^2$  with exponential orders  $N_{\pm\infty}$  as  $\phi \rightarrow \pm\infty$  such that  $N_{\pm\infty} < \sqrt{6}$  and let be  $\chi \in \mathcal{E}^2$  with exponential orders  $M_{\pm\infty}$  as  $\phi \rightarrow \pm\infty$  such that*

1.  $0 < \gamma < \frac{4}{3}$  y  $M_{\pm\infty} > (2 N_{\pm\infty} - \sqrt{6}\gamma) / (3\gamma - 4)$  or
2.  $\frac{4}{3} < \gamma < 2$  y  $M_{\pm\infty} < (2 N_{\pm\infty} - \sqrt{6}\gamma) / (3\gamma - 4)$

*Then, it is verified asymptotically that:*

$$H = \frac{1}{3t} + O(\varepsilon_V^\pm(t)), \tag{3.128}$$

$$\phi = \pm \sqrt{\frac{2}{3}} \ln \frac{t}{c} + O(t\varepsilon_V^\pm(t)), \tag{3.129}$$

$$\dot{\phi} = \pm \sqrt{\frac{2}{3}} t^{-1} + O(\varepsilon_V^\pm(t)), \tag{3.130}$$



$$\rho = \frac{b_0^2}{3} t^{-\gamma} \chi \left( -\sqrt{\frac{2}{3}} \ln \frac{t}{c} \right)^{\frac{3\gamma}{2}-2} (1 + O(t\varepsilon_V^\pm(t))), \quad (3.131)$$

where  $\varepsilon_V^\pm(t) = tV \left( \pm \sqrt{\frac{2}{3}} \ln \frac{t}{c} \right)$ .

### Sketch of the proof of theorem 25

Following the same reasoning as in [403], it is suffice to prove that all the solution, but, perhaps, a set of Lebesgue measure zero, are past asymptotic to the singular point  $P_1$  (for  $\phi \rightarrow +\infty$  or  $\phi \rightarrow -\infty$ ). Since  $x_3$  is monotonic increasing, it is enough to consider orbits in  $\Omega(x)$  for  $x$  arbitrary. Since  $\Omega(x)$  is compact and contains its past orbits; then, all the points  $\mathbf{p} \in \Omega(x)$  should have a nonempty  $\alpha$ -limit set,  $\alpha(\mathbf{p})$ . Particularly, for all the points in the physical state space  $\Omega(x)$ , theorem 22 implies that  $\alpha(\mathbf{p})$  must contain almost always a singular point with  $\varphi = 0$  ( $\phi = \pm\infty$ ). By the discussion in the section 3.4.6, each point with  $\varphi = 0$  being a limit point of the physical trajectory, must be part of the non-physical boundary  $\partial\Omega(x)$  and then must have  $x_3 = 0$ . Since  $x_3$  is monotonically increasing, the set  $\alpha(p)$  must be contained completely in the plane  $x_3 = 0$ , or namely in  $\partial\Omega(x)$ . It can be proved that the only conceivable generic past attractor are the singular points  $P_1$  in  $\pm\infty$  (the other singular points cannot be generic sources by our previous analysis in section 3.4.4.1).

## 3.5 Early-time behavior for the model including radiation

In this section we investigate the early-time dynamics of the general model by including radiation.

### 3.5.1 Normalized Variables and Dynamical System

As a difference with the previous study we introduce the new dimensionless variables

$$\sigma_1 = \phi, \sigma_2 = \frac{\dot{\phi}}{\sqrt{6}H}, \sigma_3 = \frac{\sqrt{\rho}}{\sqrt{3}H}, \sigma_4 = \frac{\sqrt{V}}{\sqrt{3}H}, \sigma_5 = \frac{\sqrt{\rho_r}}{\sqrt{3}H} \quad (3.132)$$

and the time coordinate

$$d\tau = 3H dt. \quad (3.133)$$

We considered the scalar field itself as a dynamical variable.

Using these coordinates the equations (3.2)-(3.5) recast as an autonomous system satisfying an inequality arising from the Friedmann equation (3.6). This system is given by

$$\sigma'_1 = \sqrt{\frac{2}{3}}\sigma_2 \quad (3.134)$$

$$\begin{aligned} \sigma'_2 = \sigma_2^3 + \frac{1}{6} (3\gamma\sigma_3^2 + 4\sigma_5^2 - 6) \sigma_2 - \frac{\sigma_4^2}{\sqrt{6}} \frac{d \ln V(\sigma_1)}{d\sigma_1} + \\ + \frac{(4 - 3\gamma) \sigma_3^2}{2\sqrt{6}} \frac{d \ln \chi(\sigma_1)}{d\sigma_1}, \end{aligned} \quad (3.135)$$

$$\begin{aligned} \sigma'_3 = \frac{1}{6} \sigma_3 (6\sigma_2^2 + 3\gamma(\sigma_3^2 - 1) + 4\sigma_5^2) + \\ - \frac{(4 - 3\gamma) \sigma_2 \sigma_3}{2\sqrt{6}} \frac{d \ln \chi(\sigma_1)}{d\sigma_1}, \end{aligned} \quad (3.136)$$

$$\sigma'_4 = \frac{1}{6} \sigma_4 (6\sigma_2^2 + 3\gamma\sigma_3^2 + 4\sigma_5^2) + \frac{\sqrt{6}}{6} \sigma_2 \sigma_4 \frac{d \ln V(\sigma_1)}{d\sigma_1}, \quad (3.137)$$

$$\sigma'_5 = \frac{1}{6} \sigma_5 (6\sigma_2^2 + 3\gamma\sigma_3^2 + 4\sigma_5^2 - 4). \quad (3.138)$$

The system (3.134)-(3.138) defines a flow in the phase space

$$\Sigma := \left\{ \sigma \in \mathbb{R}^5 : \sum_{j=2}^5 \sigma_j^2 = 1, \sigma_j \geq 0, j = 3, 4, 5 \right\}. \quad (3.139)$$

Now, let us proceed to investigate the topological properties of the phase space. Knowing the topological structure of the phase space allows to a better understanding of the dynamics and provides the geometrical basis for the proof of our main results.

### 3.5.2 The Topological Properties of the Phase Space

Let us define the sets  $\Sigma_0 := \{\sigma \in \Sigma : \sigma_5 = 0\}$  and  $\Sigma_+ = \{\sigma \in \Sigma : \sigma_5 > 0\}$ . By construction these sets are a partition of  $\Sigma$ .

**Proposition 11**  $\Sigma_0$  is a manifold with boundary.

**Proof.** Since  $\Sigma_0 \subset \mathbb{R}^4$  is a closed set with respect the usual topology of  $\mathbb{R}^4$ , it is a Hausdorff space equipped with a numerable basis. The rest of the proof requires the construction of a set of local charts.

Let us define the sets  $V_j := \{\sigma \in \mathbb{R}^4 : \sigma_j > 0\} \cap \Sigma_0$ ,  $j = 3, 4$ . These sets are open with respect to the induced topology in  $\Sigma_0$ .

Let be defined the projection maps

$$h_3 : V_3 \rightarrow \mathbb{H}^3, \sigma \rightarrow h_3(\sigma) = (\sigma_1, \sigma_2, \sigma_4),$$

and

$$h_4 : V_4 \rightarrow \mathbb{H}^3, \sigma \rightarrow h_4(\sigma) = (\sigma_1, \sigma_2, \sigma_3).$$

These maps satisfy  $h_j(V_j) = \mathbb{R} \times (\mathbb{H}^2 \cap \mathbb{D}^2)$ ,  $j = 3, 4$ , which are open sets of  $\mathbb{H}^3$ . Their inverses are given by

$$h_3^{-1} : \mathbb{R} \times (\mathbb{H}^2 \cap \mathbb{D}^2) \rightarrow V_3, (\sigma_1, \sigma_2, \sigma_4) \rightarrow \left( \sigma_1, \sigma_2, \sqrt{1 - \sigma_2^2 - \sigma_4^2}, \sigma_4 \right),$$

and

$$h_4^{-1} : \mathbb{R} \times (\mathbb{H}^2 \cap \mathbb{D}^2) \rightarrow V_4, (\sigma_1, \sigma_2, \sigma_3) \rightarrow \left( \sigma_1, \sigma_2, \sigma_3, \sqrt{1 - \sigma_2^2 - \sigma_3^2} \right).$$

It is clear that they are homeomorphisms.

Observe that  $\{(V_3, h_3), (V_4, h_4)\}$  does not cover the sets with  $\sigma_3 = \sigma_4 = 0$ . The construction is completed by defining the sets

$$V_1^\pm := \{\sigma \in \Sigma_0 : \sigma_2 = \pm 1, \sigma_3 = \sigma_4 = 0\}$$

which are disjoint copies of  $\mathbb{R}$ , and thus they are 1-dimensional manifolds. ■

By definition  $(V_3, h_3), (V_4, h_4)$  are topological manifolds. It can be proved that, in fact, each one is a topological manifold with boundary. The boundaries are

$$\partial V_3 = \{\sigma \in V_3 : \sigma_4 = 0\},$$

and

$$\partial V_4 = \{\sigma \in V_4 : \sigma_3 = 0\}.$$

Both are homeomorphic to  $\mathbb{R} \times (-1, 1)$ .

Let be defined  $V_1 := V_1^- \cup V_1^+$ . Observe that

$$\begin{aligned} \Sigma_0 &= (\Sigma_0 \setminus V_1) \cup V_1 = \text{Int}(\Sigma_0 \setminus V_1) \cup (\partial V_3 \cup \partial V_4) \cup V_1 \\ &= \text{Int}(\Sigma_0 \setminus V_1) \cup (\partial \Sigma_0)_1 \cup (\partial \Sigma_0)_2, \end{aligned} \quad (3.140)$$

where we have defined  $(\partial \Sigma_0)_1 = \partial V_4 \cup V_1 = \{\sigma \in \Sigma : \sigma_3 = 0\}$  and  $(\partial \Sigma_0)_2 = \partial V_3 \cup V_1 = \{\sigma \in \Sigma : \sigma_4 = 0\}$ .

From the above arguments and expression (3.140) we have:

**Remark 4** • *The interior of  $\Sigma_0$  is given by  $\text{Int} \Sigma_0 = \text{Int}(\Sigma_0 \setminus V_1)$  which is a 3-dimensional manifold (without boundary).*

- *The boundary of  $\Sigma_0$  is the union of two 2-dimensional topological manifolds with boundary given by  $(\partial \Sigma_0)_1$  and  $(\partial \Sigma_0)_2$ .*
- *$(\partial \Sigma_0)_1$  and  $(\partial \Sigma_0)_2$  share the same boundary  $V_1$  which is the union of two disjoint copies of  $\mathbb{R}$ .*

**Proposition 12**  $\Sigma_+$  is a topological manifold with boundary.

**Proof.** It is easy to check that  $\Sigma_+ \subset \mathbb{R}^5$  is a Hausdorff space equipped with a numerable basis. The rest of the proof requires the construction of a set of local charts.

Let us define the sets  $W_j := \{\sigma \in \mathbb{R}^5 : \sigma_j > 0\} \cap \Sigma_+$ ,  $j = 3, 4$ . These sets are open with respect to the induced topology in  $\Sigma_+$ .

Let us define the maps

$$g_3 : W_3 \rightarrow \mathbb{H}^4, \sigma \rightarrow g_3(\sigma) = (\sigma_1, \sigma_2, \sigma_5, \sigma_4),$$

and

$$g_4 : W_4 \rightarrow \mathbb{H}^4, \sigma \rightarrow g_4(\sigma) = (\sigma_1, \sigma_2, \sigma_5, \sigma_3).$$

These maps satisfy  $g_j(W_j) = \mathbb{R} \times (\mathbb{H}^3 \cap \mathbb{D}^3)$ ,  $j = 3, 4$  which are open set of  $\mathbb{H}^4$ . Their inverses are given by

$$\begin{aligned} g_3^{-1} : \mathbb{R} \times (\mathbb{H}^3 \cap \mathbb{D}^3) &\rightarrow W_3, \\ (\sigma_1, \sigma_2, \sigma_5, \sigma_4) &\rightarrow \left( \sigma_1, \sigma_2, \sqrt{1 - \sigma_2^2 - \sigma_4^2 - \sigma_5^2}, \sigma_4, \sigma_5 \right), \end{aligned}$$

and

$$\begin{aligned} g_4^{-1} : \mathbb{R} \times (\mathbb{H}^3 \cap \mathbb{D}^3) &\rightarrow W_4, \\ (\sigma_1, \sigma_2, \sigma_5, \sigma_3) &\rightarrow \left( \sigma_1, \sigma_2, \sigma_3, \sqrt{1 - \sigma_2^2 - \sigma_3^2 - \sigma_5^2}, \sigma_5 \right). \end{aligned}$$

It is clear that they are homeomorphism.

Observe that  $\{(W_3, g_3), (W_4, g_4)\}$  do not cover the sets with  $\sigma_3 = \sigma_4 = 0$ . The construction is completed by defining the set  $W_1 := \{\sigma \in \Sigma_+ : \sigma_1^2 + \sigma_5^2 = 1\}$ . Using the projection map  $(\sigma_1, \sigma_2, \sigma_5) \xrightarrow{g_1} (\sigma_1, \sigma_2)$  it is easily proved that  $W_1$  is homeomorphic to the open set  $\mathbb{R} \times (-1, 1)$ . ■

We have that  $W_3$  is a manifold with boundary. Its boundary is the set  $\partial W_3 := \{\sigma \in \Sigma : \sigma_3 > 0, \sigma_4 = 0, \sigma_5 > 0\}$  and its interior is the set  $\{\sigma \in \Sigma : \sigma_3 > 0, \sigma_4 > 0, \sigma_5 > 0\}$ . Also,  $W_4$  is a manifold with boundary. Its boundary is the set  $\partial W_4 := \{\sigma \in \Sigma : \sigma_3 = 0, \sigma_4 > 0, \sigma_5 > 0\}$  with same interior as  $W_3$  and  $W_1$  is a manifold without boundary which is homeomorphic to  $\mathbb{R} \times (-1, 1)$ .

Let us define the sets

$$(\partial \Sigma_+)_1 := \partial W_4 \cup W_1$$

and

$$(\partial \Sigma_+)_2 := \partial W_3 \cup W_1.$$

Following the same arguments yielding to (3.140) we get the formula

$$\Sigma_+ = \text{Int}(\Sigma_+ \setminus W_1) \cup (\partial\Sigma_+)_1 \cup (\partial\Sigma_+)_2. \quad (3.141)$$

Using the above arguments and relation (3.141) we have the following

**Remark 5** • *The interior of  $\Sigma_+$  is given by  $\text{Int}\Sigma_+ = \text{Int}(\Sigma_+ \setminus W_1)$  which is a 4-dimensional manifold (without boundary).*

- *The boundary of  $\Sigma_+$  is the union of two 3-dimensional topological manifolds with boundary given by  $(\partial\Sigma_+)_1$  and  $(\partial\Sigma_+)_2$ .*
- *$(\partial\Sigma_+)_1$  and  $(\partial\Sigma_+)_2$  share the same boundary  $W_1$  which is a 2-dimensional manifold without boundary homeomorphic to  $\mathbb{R} \times (-1, 1)$ .*

### 3.5.3 Monotonic Functions

The construction of monotonic functions in the state space is an important tool in any phase space analysis. The existence of such functions can rule out periodic orbits, homoclinic orbits, and other complex behavior in invariant sets. If so, the dynamics is dominated by singular points (and possibly, heteroclinic orbits joining it). Additionally, some global results can be obtained. In table 3.3 are displayed several monotonic functions for the flow of (3.134)-(3.138) in  $\Sigma$ .

Table 3.3: Monotonic functions for the flow of (3.134)-(3.138) in (3.139).

$Z$	$dZ/d\tau$	Invariant set	Restrictions <sup>a</sup>
$Z_1 = \left(\frac{\sigma_3}{\sigma_4}\right)^2 V(\sigma_1)\chi(\sigma_1)^{2-\frac{3\gamma}{2}}$	$-\gamma Z_1$	$\sigma_3 > 0, \sigma_4 > 0$	$\gamma \in (0, \frac{4}{3}) \cup (\frac{4}{3}, 2)$
$Z_2 = \left(\frac{\sigma_5}{\sigma_3}\right)^2 \chi(\sigma_1)^{-2+\frac{3\gamma}{2}}$	$-(\frac{4}{3} - \gamma) Z_2$	$\sigma_3 > 0, \sigma_5 > 0$	$\gamma \in (0, \frac{4}{3}) \cup (\frac{4}{3}, 2)$
$Z_3 = \left(\frac{\sigma_5}{\sigma_4}\right)^2 V(\sigma_1)$	$-\frac{4}{3} Z_3$	$\sigma_4 > 0, \sigma_5 > 0$	none
$Z_4 = \frac{\sigma_2^2}{1-\sigma_2^2}$	$-\frac{2}{3} Z_4$	$\sigma_2 \neq 0, \sigma_3 = 0$ $\sigma_4 = 0, \sigma_5 > 0$	none

<sup>a</sup> We assume the general conditions  $\chi, V \in C^3$ ,  $\chi(\sigma_1) > 0, V(\sigma_1) > 0$ .

**Remark 6** *From the definition of  $Z_1$  it follows that it is a monotonic decreasing function for the flow of (3.134)-(3.138) restricted to the invariant set  $\sigma_3 > 0, \sigma_4 > 0$ . Applying the Monotonicity Principle (theorem 18) it follows that the past attractor the flow of (3.134)-(3.138) restricted to the invariant set  $\sigma_3 > 0, \sigma_4 > 0$  is contained in the set where  $\sigma_4 = 0$  and the future attractor is contained in the set where  $\sigma_3 = 0$ .*

**Remark 7** Using the same argument follows from the definition of  $Z_2$  that the past attractor of the flow of (3.134)-(3.138) restricted to the invariant set  $\sigma_3 > 0, \sigma_5 > 0$  is contained in the invariant set where  $\sigma_3 = 0$  and the future asymptotic attractor is contained in the invariant set  $\sigma_5 = 0$  provided  $\gamma < \frac{4}{3}$ . If  $\gamma > \frac{4}{3}$  the asymptotic behavior is the reverse of the previously described.

**Remark 8** From the definition of  $Z_3$  follows that the past attractor of the flow of (3.134)-(3.138) restricted to the invariant set  $\sigma_4 > 0, \sigma_5 > 0$  is contained in the invariant set where  $\sigma_4 = 0$  and the future asymptotic attractor is contained in the invariant set  $\sigma_5 = 0$ .

**Remark 9** From the definition of  $Z_4$  follows that the past attractor of the flow of (3.134)-(3.138) restricted to the invariant set  $\sigma_2 \neq 0, \sigma_3 = \sigma_4 = 0, \sigma_5 > 0$  is contained in the invariant set where  $\sigma_2 = 0$  (i.e., where  $\sigma_5 = 1$ ) and the future asymptotic attractor is contained in the invariant set  $\sigma_2 = \pm 1$ .

### 3.5.4 Singular points with $\phi$ Bounded

Let us make a preliminary analysis of the linear stability of the singular points of the flow of (3.134)-(3.138) defined in  $\Sigma$ . It is a classic result that the linear stability of the singular points does not change under homeomorphisms.

Since  $\Sigma$  is a 4-dimensional manifold (with boundary) we will consider the projection of  $\Sigma$  in a real 4-dimensional manifold (with boundary).

Let be defined the projection map

$$\begin{aligned} \pi : \Sigma &\rightarrow \Omega \\ (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) &\rightarrow (\sigma_1, \sigma_2, \sigma_3, \sigma_5) \end{aligned} \quad (3.142)$$

where

$$\Omega := \{ \sigma \in \mathbb{R}^4 : \sigma_2^2 + \sigma_3^2 + \sigma_5^2 \leq 1, \sigma_j \geq 0, j = 3, 5 \}. \quad (3.143)$$

The flow of (3.134)-(3.138) defined on  $\Sigma$  is topologically equivalent (under  $\pi$ ) to the flow of

$$\sigma'_1 = \sqrt{\frac{2}{3}}\sigma_2 \quad (3.144)$$

$$\begin{aligned} \sigma'_2 = \sigma_2^3 + \frac{1}{6} (3\gamma\sigma_3^2 + 4\sigma_5^2 - 6) \sigma_2 - \frac{(1 - \sigma_2^2 - \sigma_3^2 - \sigma_5^2) d \ln V(\sigma_1)}{\sqrt{6} d\sigma_1} + \\ + \frac{(4 - 3\gamma) \sigma_3^2 d \ln \chi(\sigma_1)}{2\sqrt{6} d\sigma_1}, \end{aligned} \quad (3.145)$$

$$\sigma'_3 = \frac{1}{6}\sigma_3 (6\sigma_2^2 + 3\gamma(\sigma_3^2 - 1) + 4\sigma_5^2) - \frac{(4 - 3\gamma) \sigma_2 \sigma_3 d \ln \chi(\sigma_1)}{2\sqrt{6} d\sigma_1}, \quad (3.146)$$

$$\sigma'_5 = \frac{1}{6}\sigma_5 (6\sigma_2^2 + 3\gamma\sigma_3^2 + 4\sigma_5^2 - 4), \quad (3.147)$$

defined in  $\Omega$ .

Table 3.4: Location of the singular points of the flow of (3.144)-(3.147) defined in  $\Omega$ .

Label	$\sigma_1^a$	$\sigma_2$	$\sigma_3$	$\sigma_5$
$Q_1$	$\sigma_{1c} : \chi'(\sigma_{1c}) = 0$	0	1	0
$Q_2$	$\sigma_{1c} : V'(\sigma_{1c}) = 0$	0	0	0
$Q_3$	$\sigma_{1c} \in \mathbb{R}$	0	0	1

<sup>a</sup> We are assuming  $V(\sigma_{1c}) \neq 0$ ,  $\chi(\sigma_{1c}) \neq 0$  and  $\gamma \in (0, \frac{4}{3}) \cup (\frac{4}{3}, 2)$ .

The system (3.144)-(3.147) defined in  $\Omega$  admits three classes of singular points located at  $\Omega$ , denoted by  $Q_j$ ,  $j = 1, 2, 3$ . In table 3.4 are displayed the coordinates of such points. The dynamics near the singular points (and its stability properties) is dictated by the signs of the real parts of the eigenvalues of the Jacobian matrix evaluated at each singular point as follows:

1. The eigenvalues of the linearization around  $Q_1$  are  $-\Delta_2$ ,  $\gamma$ ,  $\Delta_1 \pm \sqrt{\Delta_1^2 + \Delta_2 \frac{\chi''(\phi_1)}{\chi(\phi_1)}}$ , where  $\Delta_1 = (-2 + \gamma)/4 < 0$ , and  $\Delta_2 = (4 - 3\gamma)/6$ . Then, the local stability of singular point  $P_1$  is as follows:

- (a) If  $-\frac{\Delta_1^2 \chi(\sigma_{1c})}{\Delta_2} \leq \chi''(\sigma_{1c}) < 0$  and  $0 < \gamma < \frac{4}{3}$  there exist a 3-dimensional stable manifold and a 1-dimensional unstable manifold of  $Q_1$ .
- (b) If  $0 < \gamma < \frac{4}{3}$  and  $\chi''(\sigma_{1c}) > 0$  or  $\frac{4}{3} < \gamma < 2$  and  $0 < \chi''(\sigma_{1c}) < -\frac{\Delta_1^2 \chi(\sigma_{1c})}{\Delta_2}$ , there exist a 2-dimensional stable manifold and a 2-dimensional unstable manifold of  $Q_1$ .
- (c) If  $0 < \gamma < \frac{4}{3}$  and  $\chi''(\sigma_{1c}) = 0$  there exist a 2-dimensional stable manifold, a 1-dimensional unstable manifold and a 1-dimensional center manifold of  $Q_1$ .

- (d) If  $\frac{4}{3} < \gamma < 2$  and  $\chi''(\sigma_{1c}) = 0$  there exist a 1-dimensional stable manifold, a 2-dimensional unstable manifold and a 1-dimensional center manifold of  $Q_1$ .
2. The eigenvalues of the linearization around  $Q_2$  are:  $-\frac{2}{3}, -\frac{\gamma}{2}, -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{4}{3} \frac{V''(\sigma_{1c})}{V(\sigma_{1c})}}$ . Then, the local stability of singular point  $P_2$  is as follows:
- (a) If  $V''(\sigma_{1c}) < 0$  there exists a 3-dimensional stable manifold and a 1-dimensional unstable manifold of  $Q_2$ .
- (b) If  $V''(\sigma_{1c}) = 0$  the eigenvalues are  $-1, -\frac{2}{3}, 0, -\frac{\gamma}{2}$ . Then there exists a 1-dimensional center manifold tangent to the  $\sigma_1$ -axis and a 3-dimensional stable manifold tangent to the 3-dimensional surface

$$\left\{ (\sigma_1, \sigma_2, \sigma_3, \sigma_5) \in \mathbb{R}^4 : \sigma_1 = -\sqrt{\frac{2}{3}}\sigma_2 \right\}.$$

- (c) If  $0 < V''(\sigma_{1c}) \leq \frac{3}{4}V(\sigma_{1c})$  the stable manifold of  $Q_2$  is 4-dimensional (the singular point is asymptotically stable and it is an stable node). If  $V''(\sigma_{1c}) > \frac{3}{4}V(\sigma_{1c})$  the stable manifold of  $Q_2$  is 4-dimensional (the singular point is asymptotically stable and it is an stable focus).
3. The eigenvalues of the linearization around  $Q_3$  are  $\frac{4}{3}, -\frac{1}{3}, 0, \frac{1}{6}(4 - 3\gamma)$ . Then, the local stability of singular point  $Q_3$  is as follows:

- (a) There exists a 1-dimensional center manifold tangent to the  $\sigma_1$ -axis
- (b) If  $\gamma < \frac{4}{3}$ ,  $Q_3$  has a 2-dimensional unstable manifold and a 1-dimensional stable manifold tangent to the line

$$\left\{ (\sigma_1, \sigma_2, \sigma_3, \sigma_5) \in \mathbb{R}^4 : \sigma_1 = -\sqrt{\frac{2}{3}}\sigma_2, \sigma_3 = \sigma_5 = 0 \right\}.$$

- (c) If  $\gamma > \frac{4}{3}$ ,  $Q_3$  has a 1-dimensional unstable manifold and a 2-dimensional stable manifold tangent to the 2-dimensional subspace

$$\left\{ (\sigma_1, \sigma_2, \sigma_3, \sigma_5) \in \mathbb{R}^4 : \sigma_1 = -\sqrt{\frac{2}{3}}\sigma_2, \sigma_5 = 0 \right\}.$$

Let us comment on the physical interpretation of the singular points listed above. The singular points  $Q_1$  represent matter-dominated cosmological solutions with infinite curvature ( $H \rightarrow +\infty$ ). Since  $\chi'(\sigma_{1c}) = 0$  (i.e.,  $\sigma_{1c}$  is an stationary point of the coupling function) and  $\chi'(\sigma_{1c}) \neq 0$  they are solutions with minimally coupled scalar field and negligible kinetic energy. The potential function does not influence its dynamical character.



The singular points  $Q_2$  represent a *de Sitter* cosmological solution. When one of these singular points is approached, the energy density of dark matter and the kinetic energy of the scalar field go to zero. In this case the potential energy of the scalar field becomes dominant. Hence, the universe would be expanding forever in a *de Sitter* phase.

The singular points  $Q_3$  represent radiation-dominated cosmological solutions. These solutions are very important during the radiation era.

### 3.5.5 Center manifold of $Q_2$ .

In this section we apply the center manifold theorem to examine the stability of (3.144)-(3.147) around  $Q_2$ . We exclude from the analysis the case where  $V'(\sigma_{1c}) = 0$  and  $V''(\sigma_{1c}) > 0$  (i.e.,  $V(\phi)$  has a local minimum at  $\sigma_{1c}$ ) since from the linear analysis (see point 2c in subsection 3.5.4) follows the asymptotic stability of  $Q_2$ .

Let  $V(\phi)$  and  $\chi(\phi)$  be smooth ( $C^\infty$ ) functions. The barotropic index of the fluid satisfies  $0 < \gamma < 2, \gamma \neq \frac{4}{3}$ .

**Proposition 13** *Let  $\sigma_{1c}$  such that  $V'(\sigma_{1c}) = V''(\sigma_{1c}) = 0$  and  $V^{(3)}(\sigma_{1c}) \neq 0$ , i.e.,  $\sigma_{1c}$  is an inflection point of  $V(\phi)$ . Then, the singular point  $Q_2$  of the system (3.144)-(3.147) is locally unstable.*

**Proof.** Let us consider the coordinate transformation  $(\sigma_1, \sigma_2, \sigma_3, \sigma_5) \rightarrow (\sigma_1 + \sigma_{1c}, \sigma_2, \sigma_3, \sigma_5)$ . The Taylor expansion up to third order of the system arising from (3.144)-(3.147) under such coordinate transformation around the origin reads

$$\begin{aligned}\sigma'_1 &= \sqrt{\frac{2}{3}}\sigma_2 + \mathcal{O}(3), \\ \sigma'_2 &= -\sigma_2 - \frac{\sigma_1^2}{2\sqrt{6}} \frac{V^{(3)}(\sigma_{1c})}{V(\sigma_{1c})} + \frac{(4-3\gamma)\sigma_3^2}{2\sqrt{6}} \frac{\chi'(\sigma_{1c})}{\chi(\sigma_{1c})} + \mathcal{O}(3), \\ \sigma'_3 &= -\frac{\gamma}{2}\sigma_3 - \frac{(4-3\gamma)\sigma_2\sigma_3}{2\sqrt{6}} \frac{\chi'(\sigma_{1c})}{\chi(\sigma_{1c})} + \mathcal{O}(3), \\ \sigma'_5 &= -\frac{2}{3}\sigma_5 + \mathcal{O}(3).\end{aligned}\tag{3.148}$$

The change of coordinates

$$\sigma_1 \rightarrow x - \sqrt{\frac{2}{3}}y_2, \sigma_2 \rightarrow y_2, \sigma_3 \rightarrow y_3, \sigma_5 \rightarrow y_1$$

allows to reduce the linear part of (3.148) to its real Jordan form

$$\begin{aligned}
x' &= -\frac{V^{(3)}(\sigma_{1c})x^2}{6V(\sigma_{1c})} + \frac{\sqrt{\frac{2}{3}}y_2V^{(3)}(\sigma_{1c})x}{3V(\sigma_{1c})} + \frac{2y_3^2\chi'(\sigma_{1c})}{3\chi(\sigma_{1c})} - \frac{y_3^2\gamma\chi'(\sigma_{1c})}{2\chi(\sigma_{1c})} + \\
&\quad - \frac{y_2^2V^{(3)}(\sigma_{1c})}{9V(\sigma_{1c})} + \mathcal{O}(3), \\
y_1' &= -\frac{2y_1}{3} + \mathcal{O}(3), \\
y_2' &= -\frac{V^{(3)}(\sigma_{1c})x^2}{2\sqrt{6}V(\sigma_{1c})} + \frac{y_2V^{(3)}(\sigma_{1c})x}{3V(\sigma_{1c})} - y_2 + \frac{\sqrt{\frac{2}{3}}y_3^2\chi'(\sigma_{1c})}{\chi(\sigma_{1c})} + \\
&\quad - \frac{\sqrt{\frac{3}{2}}y_3^2\gamma\chi'(\sigma_{1c})}{2\chi(\sigma_{1c})} - \frac{y_2^2V^{(3)}(\sigma_{1c})}{3\sqrt{6}V(\sigma_{1c})} + \mathcal{O}(3), \\
y_3' &= -\frac{y_3\gamma}{3} + \frac{\sqrt{\frac{3}{2}}y_2y_3\chi'(\sigma_{1c})\gamma}{2\chi(\sigma_{1c})} - \frac{\sqrt{\frac{2}{3}}y_2y_3\chi'(\sigma_{1c})}{\chi(\sigma_{1c})} + \mathcal{O}(3).
\end{aligned} \tag{3.149}$$

Hence, the system (3.149) is written in diagonal form

$$\begin{aligned}
x' &= Cx + f(x, \mathbf{y}) \\
\mathbf{y}' &= P\mathbf{y} + \mathbf{g}(x, \mathbf{y}),
\end{aligned} \tag{3.150}$$

where  $(x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^3$ ,  $C$  is the zero  $1 \times 1$  matrix,  $P$  is a  $3 \times 3$  matrix with negative eigenvalues and  $f, \mathbf{g}$  vanish at  $\mathbf{0}$  and have vanishing derivatives at  $\mathbf{0}$ . The center manifold theorem 13 asserts that there exists a 1-dimensional invariant local center manifold  $W^c(\mathbf{0})$  of (3.150) tangent to the center subspace (the  $\mathbf{y} = \mathbf{0}$  space) at  $\mathbf{0}$ . Moreover,  $W^c(\mathbf{0})$  can be represented as

$$W^c(\mathbf{0}) = \{(x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^3 : \mathbf{y} = \mathbf{h}(x), |x| < \delta\}; \mathbf{h}(0) = \mathbf{0}, D\mathbf{h}(0) = \mathbf{0},$$

for  $\delta$  sufficiently small (see definition 13). The restriction of (3.150) to the center manifold is (see definition 2.36)

$$x' = f(x, \mathbf{h}(x)). \tag{3.151}$$

According to Theorem 14, if the origin  $x = 0$  of (3.151) is stable (asymptotically stable) (unstable) then the origin of (3.150) is also stable (asymptotically stable) (unstable). Therefore, we have to find the local center manifold, i.e., the problem reduces to the computation of  $\mathbf{h}(x)$ .

Substituting  $\mathbf{y} = \mathbf{h}(x)$  in the second component of (3.200) and using the chain rule,  $\mathbf{y}' = D\mathbf{h}(x)x'$ , one can show that the function  $\mathbf{h}(x)$  that defines the local center manifold satisfies

$$D\mathbf{h}(x)[f(x, \mathbf{h}(x))] - P\mathbf{h}(x) - \mathbf{g}(x, \mathbf{h}(x)) = 0. \tag{3.152}$$

According to Theorem 15, equation (3.152) can be solved approximately by using an approximation of  $\mathbf{h}(x)$  by a Taylor series at  $x = 0$ . Since  $\mathbf{h}(0) = \mathbf{0}$  and  $D\mathbf{h}(0) = \mathbf{0}$ , it is obvious that  $\mathbf{h}(x)$  commences with quadratic terms. We substitute

$$\mathbf{h}(x) =: \begin{bmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \end{bmatrix} = \begin{bmatrix} a_1x^2 + a_2x^3 + O(x^4) \\ b_1x^2 + b_2x^3 + O(x^4) \\ c_1x^2 + c_2x^3 + O(x^4) \end{bmatrix}$$

into (3.152) and set the coefficients of like powers of  $x$  equal to zero to find the unknowns  $a_1, b_1, c_1, \dots$ . It is straightforward to find that

$$(a_1, a_2, b_1, b_2, c_1, c_2) = \left( 0, 0, -\frac{V^{(3)}(\sigma_{1c})}{2\sqrt{6}V(\sigma_{1c})}, -\frac{V^{(3)}(\sigma_{1c})^2}{3\sqrt{6}V(\sigma_{1c})^2}, 0, 0 \right).$$

Thus, the restriction of the vector field to the center manifold of the origin is given by

$$x' = -\frac{V^{(3)}(\sigma_{1c})x^2}{6V(\sigma_{1c})} - \frac{V^{(3)}(\sigma_{1c})^2x^3}{18V(\sigma_{1c})^2} - \frac{V^{(3)}(\sigma_{1c})^3x^4}{24V(\sigma_{1c})^3} + \mathcal{O}(x^5). \quad (3.153)$$

This is a gradient like vector field whose potential function has an inflection point at the origin irrespective the sign of the ratio  $\frac{V^{(3)}(\sigma_{1c})}{V(\sigma_{1c})}$ . From this follows that, under the conditions discussed here, the origin of coordinates is locally unstable (of saddle type) for the flow of the original system (3.148).

■

**Proposition 14** *Let the function  $V(\phi)$  to have a degenerate local minimum (maximum) at  $\sigma_{1c}$  of order  $n = 2$ . Then, the singular point  $Q_2$  of the system (3.144)-(3.147) is locally asymptotically stable (unstable).*

**Proof.** By considering the coordinate transformation  $(\sigma_1, \sigma_2, \sigma_3, \sigma_5) \rightarrow (\sigma_1 + \sigma_{1c}, \sigma_2, \sigma_3, \sigma_5)$ . By Taylor expanding up to fourth order the system (3.144)-(3.147) under such coordinate transformation around the origin reads

$$\begin{aligned}
\sigma'_1 &= \sqrt{\frac{2}{3}}\sigma_2 + \mathcal{O}(4), \\
\sigma'_2 &= -\frac{V^{(4)}(\sigma_{1c})\sigma_1^3}{6\sqrt{6}V(\sigma_{1c})} + \frac{(3\gamma-4)\sigma_3^2\chi'(\sigma_{1c})^2\sigma_1}{2\sqrt{6}\chi(\sigma_{1c})^2} + \frac{(4-3\gamma)\sigma_3^2\chi''(\sigma_{1c})\sigma_1}{2\sqrt{6}\chi(\sigma_{1c})} + \\
&\quad + \sigma_2^3 - \sigma_2 + \frac{1}{6}\sigma_2(3\gamma\sigma_3^2 + 4\sigma_5^2) + \frac{(4-3\gamma)\sigma_3^2\chi'(\sigma_{1c})}{2\sqrt{6}\chi(\sigma_{1c})} + \mathcal{O}(4), \\
\sigma'_3 &= \frac{(4-3\gamma)\sigma_1\sigma_2\sigma_3\chi'(\sigma_{1c})^2}{2\sqrt{6}\chi(\sigma_{1c})^2} + \frac{(3\gamma-4)\sigma_2\sigma_3\chi'(\sigma_{1c})}{2\sqrt{6}\chi(\sigma_{1c})} - \frac{\gamma\sigma_3}{2} + \\
&\quad + \frac{1}{6}\sigma_3(6\sigma_2^2 + 3\gamma\sigma_3^2 + 4\sigma_5^2) + \frac{(3\gamma-4)\sigma_1\sigma_2\sigma_3\chi''(\sigma_{1c})}{2\sqrt{6}\chi(\sigma_{1c})} + \mathcal{O}(4), \\
\sigma'_5 &= \frac{1}{6}\sigma_5(6\sigma_2^2 + 3\gamma\sigma_3^2 + 4\sigma_5^2) - \frac{2\sigma_5}{3} + \mathcal{O}(4). \tag{3.154}
\end{aligned}$$

The change of coordinates

$$\sigma_1 \rightarrow x - \sqrt{\frac{2}{3}}y_2, \sigma_2 \rightarrow y_2, \sigma_3 \rightarrow y_3, \sigma_5 \rightarrow y_1$$

allows to reduce the linear part of (3.154) to its real Jordan form

$$\begin{aligned}
x' &= -\frac{V^{(4)}(\sigma_{1c})x^3}{18V(\sigma_{1c})} + \frac{y_2V^{(4)}(\sigma_{1c})x^2}{3\sqrt{6}V(\sigma_{1c})} - \frac{2y_3^2\chi'(\sigma_{1c})^2x}{3\chi(\sigma_{1c})^2} + \frac{y_3^2\gamma\chi'(\sigma_{1c})^2x}{2\chi(\sigma_{1c})^2} \\
&\quad + \frac{2y_3^2\chi''(\sigma_{1c})x}{3\chi(\sigma_{1c})} - \frac{y_3^2\gamma\chi''(\sigma_{1c})x}{2\chi(\sigma_{1c})} - \frac{y_2^2V^{(4)}(\sigma_{1c})x}{9V(\sigma_{1c})} + \sqrt{\frac{2}{3}}y_2^3 \\
&\quad + \frac{2\sqrt{\frac{2}{3}}y_2y_3^2\chi'(\sigma_{1c})^2}{3\chi(\sigma_{1c})^2} - \frac{y_2y_3^2\gamma\chi'(\sigma_{1c})^2}{\sqrt{6}\chi(\sigma_{1c})^2} + \frac{2}{3}\sqrt{\frac{2}{3}}y_1^2y_2 + \frac{y_2^3\gamma}{\sqrt{6}} + \frac{2y_3^2\chi'(\sigma_{1c})}{3\chi(\sigma_{1c})} \\
&\quad - \frac{y_3^2\gamma\chi'(\sigma_{1c})}{2\chi(\sigma_{1c})} - \frac{2\sqrt{\frac{2}{3}}y_2y_3^2\chi''(\sigma_{1c})}{3\chi(\sigma_{1c})} + \frac{y_2y_3^2\gamma\chi''(\sigma_{1c})}{\sqrt{6}\chi(\sigma_{1c})} + \frac{\sqrt{\frac{2}{3}}y_2^3V^{(4)}(\sigma_{1c})}{27V(\sigma_{1c})} + \mathcal{O}(4),
\end{aligned}$$

$$y'_1 = \frac{2y_1^3}{3} + y_2^2y_1 + \frac{1}{2}y_3^2\gamma y_1 - \frac{2y_1}{3} + \mathcal{O}(4),$$

$$\begin{aligned}
y_2' = & -\frac{V^{(4)}(\sigma_{1c})x^3}{6\sqrt{6}V(\sigma_{1c})} + \frac{y_2V^{(4)}(\sigma_{1c})x^2}{6V(\sigma_{1c})} - \frac{\sqrt{\frac{2}{3}}y_3^2\chi'(\sigma_{1c})^2x}{\chi(\sigma_{1c})^2} + \\
& + \frac{\sqrt{\frac{3}{2}}y_3^2\gamma\chi'(\sigma_{1c})^2x}{2\chi(\sigma_{1c})^2} + \frac{\sqrt{\frac{2}{3}}y_3^2\chi''(\sigma_{1c})x}{\chi(\sigma_{1c})} - \frac{\sqrt{\frac{3}{2}}y_3^2\gamma\chi''(\sigma_{1c})x}{2\chi(\sigma_{1c})} - \frac{y_2^2V^{(4)}(\sigma_{1c})x}{3\sqrt{6}V(\sigma_{1c})} + \\
& + y_2^3 + \frac{2y_2y_3^2\chi'(\sigma_{1c})^2}{3\chi(\sigma_{1c})^2} - \frac{y_2y_3^2\gamma\chi'(\sigma_{1c})^2}{2\chi(\sigma_{1c})^2} + \frac{2y_1^2y_2}{3} - y_2 + \frac{y_2^3\gamma}{2} + \frac{\sqrt{\frac{2}{3}}y_3^2\chi'(\sigma_{1c})}{\chi(\sigma_{1c})} \\
& - \frac{\sqrt{\frac{3}{2}}y_3^2\gamma\chi'(\sigma_{1c})}{2\chi(\sigma_{1c})} - \frac{2y_2y_3^2\chi''(\sigma_{1c})}{3\chi(\sigma_{1c})} + \frac{y_2y_3^2\gamma\chi''(\sigma_{1c})}{2\chi(\sigma_{1c})} + \frac{y_2^3V^{(4)}(\sigma_{1c})}{27V(\sigma_{1c})} + \mathcal{O}(4), \\
\\
y_3' = & \frac{\gamma y_3^3}{2} + \frac{2y_1^2y_3}{3} + y_2^2y_3 - \frac{2y_2^2\chi'(\sigma_{1c})^2y_3}{3\chi(\sigma_{1c})^2} + \frac{\sqrt{\frac{2}{3}}xy_2\chi'(\sigma_{1c})^2y_3}{\chi(\sigma_{1c})^2} \\
& + \frac{y_2^2\gamma\chi'(\sigma_{1c})^2y_3}{2\chi(\sigma_{1c})^2} - \frac{\sqrt{\frac{3}{2}}xy_2\gamma\chi'(\sigma_{1c})^2y_3}{2\chi(\sigma_{1c})^2} - \frac{\gamma y_3}{2} - \frac{\sqrt{\frac{2}{3}}y_2\chi'(\sigma_{1c})y_3}{\chi(\sigma_{1c})} \\
& + \frac{\sqrt{\frac{3}{2}}y_2\gamma\chi'(\sigma_{1c})y_3}{2\chi(\sigma_{1c})} + \frac{2y_2^2\chi''(\sigma_{1c})y_3}{3\chi(\sigma_{1c})} - \frac{\sqrt{\frac{2}{3}}xy_2\chi''(\sigma_{1c})y_3}{\chi(\sigma_{1c})} \\
& - \frac{y_2^2\gamma\chi''(\sigma_{1c})y_3}{2\chi(\sigma_{1c})} + \frac{\sqrt{\frac{3}{2}}xy_2\gamma\chi''(\sigma_{1c})y_3}{2\chi(\sigma_{1c})} + \mathcal{O}(4). \tag{3.155}
\end{aligned}$$

Then, we proceed to the calculation of the center manifold. The procedure is fairly systematic and since we present it completely in the previous analysis we consider do not repeat it here. Instead, we present the relevant calculations. We obtain  $a_1 = 0, a_2 = 0, b_1 = 0, b_2 = -\frac{V^{(4)}(\sigma_{1c})}{6\sqrt{6}V(\sigma_{1c})}, c_1 = 0, c_2 = 0$  for the Taylor expansion coefficients of

$$\mathbf{h}(x) =: \begin{bmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \end{bmatrix} = \begin{bmatrix} a_1x^2 + a_2x^3 + \mathcal{O}(x^4) \\ b_1x^2 + b_2x^3 + \mathcal{O}(x^4) \\ c_1x^2 + c_2x^3 + \mathcal{O}(x^4) \end{bmatrix}.$$

By substituting this values of the unknowns  $a_1, b_1, c_1, \dots$  we obtain that the dynamics of the center manifold in given by equation

$$x' = -\frac{V^{(4)}(\sigma_{1c})x^3}{18V(\sigma_{1c})} - \frac{V^{(4)}(\sigma_{1c})^2x^5}{108V(\sigma_{1c})^2} + \mathcal{O}(x^6). \tag{3.156}$$

This is a gradient-like vector field  $x' = -\nabla U(x)$  whose potential is given by

$$U(x) = \frac{V^{(4)}(\sigma_{1c})^2x^6}{648V(\sigma_{1c})^2} + \frac{V^{(4)}(\sigma_{1c})x^4}{72V(\sigma_{1c})}.$$

Since  $V(\sigma_{1c}) > 0$ ,  $U(x)$  has a degenerate local minimum (maximum) at the origin for  $V^{(4)}(\sigma_{1c}) < 0$  ( $V^{(4)}(\sigma_{1c}) > 0$ ). Thus, follows the desired result. ■ **Comment.** This analysis is in agreement with the result of proposition 7 which states that if  $V(\sigma_1)$  have a strict degenerate local minimum at  $\sigma_{1c}$  with  $V(\sigma_{1c}) > 0$ , then  $Q_2 := (\sigma_{1c}, 0, 0, 0)$  is asymptotically stable.

Now, let us formulate generalization of theorem 22 (see also theorem 3.2 of [136] p. 8.)

**Theorem 26** Assume that  $\chi(\phi)$  and  $V(\phi)$  are positive functions of class  $C^3$ , such that  $\chi$  has at most a finite number of stationary points and does not tend to zero in any compact set of  $\mathbb{R}$ . Let  $\gamma \in (0, \frac{4}{3}) \cup (\frac{4}{3}, 2)$  and let  $p$  be a point in  $\Sigma_+$ . Let  $O^-(p)$  be the past orbit of  $p$  under the flow of (3.134)-(3.138) restricted to  $\Sigma_+$ . Then,  $\phi$  is always unbounded on  $O^-(p)$  for almost all  $p$ .

### Proof of theorem 26

In order to prove the theorem it is sufficient to consider interior points of  $\Sigma_+$ . Also, in order to apply results concerning future attractors and  $\omega$ -limit sets we perform the time reversal  $\tau \rightarrow -\tau$ . Thus we get the system

$$\sigma'_1 = -\sqrt{\frac{2}{3}}\sigma_2 \quad (3.157)$$

$$\begin{aligned} \sigma'_2 = & -\sigma_2^3 - \frac{1}{6}(3\gamma\sigma_3^2 + 4\sigma_5^2 - 6)\sigma_2 + \frac{\sigma_4^2}{\sqrt{6}} \frac{d \ln V(\sigma_1)}{d\sigma_1} \\ & - \frac{(4-3\gamma)\sigma_3^2}{2\sqrt{6}} \frac{d \ln \chi(\sigma_1)}{d\sigma_1}, \end{aligned} \quad (3.158)$$

$$\begin{aligned} \sigma'_3 = & -\frac{1}{6}\sigma_3(6\sigma_2^2 + 3\gamma(\sigma_3^2 - 1) + 4\sigma_5^2) \\ & + \frac{(4-3\gamma)\sigma_2\sigma_3}{2\sqrt{6}} \frac{d \ln \chi(\sigma_1)}{d\sigma_1}, \end{aligned} \quad (3.159)$$

$$\sigma'_4 = -\frac{1}{6}\sigma_4(6\sigma_2^2 + 3\gamma\sigma_3^2 + 4\sigma_5^2) - \frac{\sqrt{6}}{6}\sigma_2\sigma_4 \frac{d \ln V(\sigma_1)}{d\sigma_1}, \quad (3.160)$$

$$\sigma'_5 = -\frac{1}{6}\sigma_5(6\sigma_2^2 + 3\gamma\sigma_3^2 + 4\sigma_5^2 - 4). \quad (3.161)$$

where the prime denotes now derivative with respect to  $-\tau$ .

Let  $p_0 := (\sigma_{10}, \sigma_{20}, \sigma_{30}, \sigma_{40}, \sigma_{50}) \in \text{Int}\Sigma_+$  such that there exist a real positive number  $K$  with  $|\sigma_1| < K$  for all  $p := (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \in O^+(p_0)$ , where  $O^+(p_0)$  denotes the positive (future) orbit for the flow of (3.157)-(3.161). Then, for all  $p \in O^+(p_0)$  we have

$$-1 \leq \sigma_2 \leq 1, 0 \leq \sigma_3 \leq 1, 0 \leq \sigma_4 \leq 1, 0 \leq \sigma_5 \leq 1.$$

Hence  $O^+(p_0)$  is contained in a compact set of (the closure of)  $\Sigma_+$ .

Since  $O^+(p_0)$  is a positive invariant set, then using proposition 3 we ensure the existence of a non empty, closed, connected and invariant  $\omega$ -limit of  $p_0$  denoted by  $\omega(p_0)$ .

First we demonstrate by contradiction that that  $\sigma_3$  and  $\sigma_4$  cannot be simultaneously zero at  $\omega(p_0)$ . Suppose that  $\omega(p_0)$  is contained in the set where  $\sigma_3 = \sigma_4 = 0$ . Let us define the function  $M_1 = Z_4^{-1}$  (see table 3.3 for the definition of  $Z_4$ ) defined in the invariant set  $\sigma_3 = \sigma_4 = 0, 0 < \sigma_5 < 1$ . From the definition of  $M_1$  and applying the Monotonicity principle (theorem 18) follows that the future asymptotic attractor of the flow of (3.157)-(3.161) restricted to the invariant set  $\sigma_3 = \sigma_4 = 0$  is contained in the invariant set  $\sigma_2 = \pm 1$ . Thus, from (3.157) follows that  $\phi \rightarrow \mp\infty$  as  $\omega(p_0)$  is approached, a contradiction.

Second, let be defined in  $Int\Sigma_+$  the function  $M_2 = Z_1^{-1}$  (see table 3.3 for the definition of  $Z_1$ ). The derivative of  $M_2$  along any orbit of (3.157)-(3.161) is given by  $M_2' = -\gamma M_2$ . Then  $M_2$  is a  $C^3$  monotonic decreasing function for the flow taking values in the interval  $(0, +\infty)$ . Since  $\sigma_3$  and  $\sigma_4$  cannot tend to zero simultaneously in  $\omega(p)$  for  $p \in O^+(p_0)$ , then the function  $M_2$  tends asymptotically to a well defined limit. By construction  $M_2(p) \rightarrow 0$  if and only if  $p \rightarrow q$  with  $q$  satisfying  $\sigma_4 = 0$  (we are using here the condition that  $\chi(\phi)$  does not tend to zero in any compact of  $\mathbb{R}$ ) and  $M_2(p) \rightarrow +\infty$  if and only if  $p \rightarrow q$  with  $q$  satisfying  $\sigma_3 = 0$ . Thus, applying the Monotonicity principle (theorem 18) follows that

$$\omega(p_0) \subset \{p \in \Sigma_+ : |\sigma_1| < K, \sigma_3 > 0, \sigma_4 = 0\} = S_1.$$

Let  $q_0 \in \omega(p_0)$ . By the invariance of the  $\omega$ -limit set follows that  $\omega(q_0) = \omega(p_0)$ .

Observe that  $g_3(S_1) = \{\sigma \in W_3 : |\sigma_1| < K, \sigma_4 = 0\}$  where  $g_3$  is defined in the proof of proposition 12.

Let us define the projection map

$$g : (\sigma_1, \sigma_2, \sigma_3, \sigma_5) \rightarrow (\sigma_1, \sigma_2, \sigma_5)$$

and let  $\sigma_0 = g \circ g_3(q_0)$  then the flow of (3.157)-(3.161) in a neighborhood of  $q_0$  contained in  $S_1$ , is topologically equivalent to the flow of

$$\begin{aligned} \sigma_1' &= -\sqrt{\frac{2}{3}}\sigma_2, \\ \sigma_2' &= \frac{1}{2}(1 - \sigma_2^2) \left( (2 - \gamma)\sigma_2 - \frac{(4 - 3\gamma)}{\sqrt{6}} \frac{d \ln \chi(\sigma_1)}{d\sigma_1} \right) + \\ &\quad - \frac{(4 - 3\gamma)\sigma_5^2}{6} \left( \sigma_2 - \frac{\sqrt{6}}{2} \frac{d \ln \chi(\sigma_1)}{d\sigma_1} \right), \\ \sigma_5' &= -\frac{1}{6}\sigma_5 (3(2 - \gamma)\sigma_2^2 - (4 - 3\gamma)(1 - \sigma_5^2)), \end{aligned} \tag{3.162}$$

in a neighborhood of  $\sigma_0$  contained in

$$S = \{(\sigma_1, \sigma_2, \sigma_5) : -K < \sigma_1 < K, \sigma_2^2 + \sigma_5^2 < 1, \sigma_5 > 0\}.$$

Since the vector field is  $C^2$  we can extend the flow of (3.162) to the closure of  $S$  (denoted by  $\bar{S}$ ).

Let us investigate all possible compact, non empty, and connected invariant sets of (3.167)-(3.168) located in the closure of  $S$  (these ones can be candidates to the  $\omega$ -limit  $\omega(\sigma_0)$ ).

Let us consider two cases:

i)  $0 < \gamma < \frac{4}{3}$ . Let be defined in  $S$ , the function

$$M_3(\sigma) = \frac{(1 - \sigma_2^2 - \sigma_5^2)^2 \chi(\sigma_1)^{4-3\gamma}}{\sigma_5^4}. \quad (3.163)$$

The derivative of  $M_3$  through an arbitrary orbit of (3.162) is given by

$$M'_3 = -2 \left( \frac{4}{3} - \gamma \right) M_3.$$

Then  $M_3$  is a  $C^3$  monotonic decreasing function for the flow taking values in the interval  $(0, +\infty)$ . By construction  $M_3(p) \rightarrow 0$  if and only if  $p \rightarrow q$  with  $q$  satisfying  $\sigma_2^2 + \sigma_5^2 = 1$  (since  $\chi$  does tends to zero in  $[-K, K]$ ) and  $M_3(p) \rightarrow +\infty$  if and only if  $p \rightarrow q$  with  $q$  satisfying  $\sigma_5 = 0$ . Thus, applying the Monotonicity principle (theorem 18) follows that

$$\omega(\sigma_0) \subset \{\sigma \in \bar{S} \setminus S : \sigma_2^2 + \sigma_5^2 = 1\}.$$

Let  $q_0 \in \omega(\sigma_0)$ . By the invariance of the  $\omega$ -limit follows that  $\omega(\sigma_0) = \omega(q_0)$ .

Let us define the projection map

$$g' : (\sigma_1, \sigma_2, \sigma_5) \rightarrow (\sigma_1, \sigma_2)$$

and let  $\sigma'_0 = g'(q_0)$  then the flow of (3.162) in a neighborhood of  $q_0$  contained in  $S$ , is topologically equivalent to the flow of

$$\begin{aligned} \sigma'_1 &= -\sqrt{\frac{2}{3}}\sigma_2, \\ \sigma'_2 &= \frac{1}{3}\sigma_2(1 - \sigma_2)(1 + \sigma_2). \end{aligned} \quad (3.164)$$

in a neighborhood of  $\sigma'_0$  (contained in  $S' := (-K, K) \times (-1, 1)$ ).



Let be defined in  $S'$  the function

$$M_4(\sigma) = \frac{1 - \sigma_2^2}{\sigma_2^2}$$

which satisfies  $M_4' = -\frac{2}{3}M_4$  along an arbitrary orbit of (3.164). Thus  $M_4$  is a  $C^3$  monotonic decreasing function in  $S'$ . Applying the monotonicity principle (18) follows that

$$\omega(\sigma'_0) \subset \{\sigma \in \bar{S}' \setminus S' : \sigma_2^2 = 1\}.$$

Thus  $\omega(\sigma'_0)$  is contained in one of the invariant sets of (3.167)-(3.168) given by  $\sigma_2 = \pm 1$  but this would imply the divergence of  $\phi$ . A contradiction.

ii)  $\frac{4}{3} < \gamma < 2$ . Let be defined in  $S$ , the function

$$M_5(\sigma) = \frac{\sigma_5^2 \chi(\sigma_1)^{3\gamma-4}}{(1 - \sigma_2^2 - \sigma_5^2)^2}. \quad (3.165)$$

The derivative of  $M_5$  through an arbitrary orbit of (3.162) is given by

$$M_5' = -2 \left( \gamma - \frac{4}{3} \right) M_5.$$

Then  $M_5$  is a  $C^3$  monotonic decreasing function for the flow taking values in the interval  $(0, +\infty)$ . By construction  $M_5(p) \rightarrow 0$  if and only if  $p \rightarrow q$  with  $q$  satisfying  $\sigma_5 = 0$  (since  $\chi$  does not tends to zero in  $[-K, K]$ ) and  $M_5(p) \rightarrow +\infty$  if and only if  $p \rightarrow q$  with  $q$  satisfying  $\sigma_2^2 + \sigma_5^2 = 1$ . Applying the Monotonicity principle (theorem 18) follows that

$$\omega(\sigma_0) \subset \{\sigma \in \bar{S} \setminus S : \sigma_5 = 0\}.$$

Let  $q_0 \in \omega(\sigma_0)$ . By the invariance of the  $\omega$ -limit follows that  $\omega(\sigma_0) = \omega(q_0)$ .

Let us define the projection map

$$h : (\sigma_1, \sigma_2, \sigma_5) \rightarrow (\sigma_1, \sigma_2). \quad (3.166)$$

Let  $\sigma'_0 = h(q_0)$ . Then, then the flow of (3.162) in a neighborhood of  $q_0$  contained in  $S$ , is topologically equivalent to the flow of

$$\sigma'_1 = -\sqrt{\frac{2}{3}}\sigma_2, \quad (3.167)$$

$$\sigma'_2 = \frac{1}{2}(1 - \sigma_2^2) \left( (2 - \gamma)\sigma_2 - \frac{(4 - 3\gamma)}{\sqrt{6}} \frac{d \ln \chi(\sigma_1)}{d\sigma_1} \right), \quad (3.168)$$

in a neighborhood of  $\sigma'_0$  (contained in  $S'$ ).

Let us investigate the possible compact invariant sets of (3.167)-(3.168) located in the closure of  $S'$  which can be candidates to the  $\omega$ -limit  $\omega(\sigma'_0)$ .

First  $\omega(\sigma'_0)$  cannot be contained in the invariant sets of (3.167)-(3.168) given by  $\sigma_2 = \pm 1$  because this would imply the divergence of  $\phi$ , a contradiction. Second, combining the results of the Poincaré-Bendixon Theorem (theorem 19) and Dulac's criterion (theorem 21) with  $B(\xi) = (1 - \sigma_2^2)^{-1}$  follows that the only possible compact invariant sets are the singular points with  $\sigma_1$  bounded (or heteroclinic orbits joining such singular points).

Let us consider  $\chi(\sigma_1)$  other than exponential.<sup>10</sup> In this case the system (3.167)-(3.168) admits a (possibly empty) family of singular points

$$Q := \{(q_1, 0) \in [-K, K] \times [-1, 1] : \chi'(q_1) = 0\}.$$

If  $Q = \emptyset$ , i.e.,  $\chi'(q_1) \neq 0$  for all  $|q_1| < K$ , then the future orbit  $O^+(\sigma_0)$  tends to a point with  $\sigma_1 = \pm 1$ . From this follows that  $\phi$  is unbounded (a contradiction) and the proof is done.

Let us assume that  $Q \neq \emptyset$ . Let  $q \in Q$ . The eigenvalues of the Jacobian matrix  $\frac{\partial f^i}{\partial \sigma_j}(q)$ ,  $i, j = 1, 2$  are  $\mu^\pm = \Delta_1 \pm \sqrt{\Delta_1^2 + \Delta_2 \frac{\chi''(q)}{\chi(q)}}$ , where  $\Delta_1 = \frac{2-\gamma}{4} > 0$ ,  $\Delta_2 = \frac{4-3\gamma}{6}$ . Hence, at least one of its associated eigenvalues has positive real part. Let be defined the sets  $Q^\pm = \{q \in Q : \pm \chi''(q) > 0\}$  and  $Q^0 = \{q \in Q : \chi(q) = 0\}$ . At least one of these sets is not empty. Let be define  $R = \{p \in [-K, K] \times [-1, 1] : \lim_{\tau \rightarrow \infty} \mathbf{g}^\tau(p) = q\}$ . There are the following cases

- $q \in Q^+$ ,  $\frac{4}{3} < \gamma < 2$ , then  $\mathcal{E}^u(q)$  is 2-dimensional implying  $R = \emptyset$ .
- $q \in Q^-$ ,  $\frac{4}{3} < \gamma < 2$ , then  $\mathcal{E}^u(q)$  is 1-dimensional and  $\mathcal{E}^s(q)$  is 1-dimensional. Then  $R \subset N$ ,  $leb(N) = 0$ .
- $q \in Q^0$ , then  $\mathcal{E}^c(q)$  is 1-dimensional and  $\mathcal{E}^u(q)$  is 1-dimensional in such way that  $R \subset \mathcal{E}^c(q)$ ,  $leb(\mathcal{E}^c(q)) = 0$ .

Therefore, all solutions future asymptotic to  $q$  (and then with  $\phi$  bounded towards the future) must lie on an stable manifold or center manifold of dimension  $r < 2$ , and then contained in a subset of  $[-K, K] \times [-1, 1]$  with zero Lebesgue measure. Since there are at most a finite number of such  $q$  the result of the theorem follows.

■

Theorem 26 allow us to conclude that in order to investigate the generic asymptotic behavior of the system (3.134)-(3.138) restricted to  $\Sigma_+$  it is sufficient to study the region where  $\phi = \pm\infty$ .

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<sup>10</sup>As we will see next in section 3.6.1, the following analysis applies also to exponential coupling functions.

### 3.5.6 Analysis in the Limit $\phi \rightarrow \infty$ .

In this section we will investigate the flow as  $\phi \rightarrow \infty$  following the nomenclature and formalism introduced in [403] (see also [414]). Analogous results hold as  $\phi \rightarrow -\infty$ .

By assuming that  $V, \chi \in \mathcal{E}_+^3$ , with exponential orders  $N$  and  $M$  respectively, we can define a dynamical system well suited to investigate the dynamics near the initial singularity. We will investigate the singular points therein. Particularly those representing scaling solutions and those associated with the initial singularity.

Let  $\Sigma_\epsilon = \{(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \in \Sigma : \sigma_1 > \epsilon^{-1}\}$  where  $\epsilon$  is any positive constant which is chosen sufficiently small so as to avoid any points where  $V$  or  $\chi = 0$ , thereby ensuring that  $\overline{W}_V(\varphi)$  and  $\overline{W}_\chi(\varphi)$  are well-defined.<sup>11</sup>

Let be defined the projection map

$$\begin{aligned} \pi_1 : \Sigma_\epsilon &\rightarrow \Omega_\epsilon \\ (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) &\rightarrow (\sigma_1, \sigma_2, \sigma_4, \sigma_5) \end{aligned} \quad (3.169)$$

where

$$\Omega_\epsilon := \{\sigma \in \mathbb{R}^4 : \sigma_1 > \epsilon^{-1}, \sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1, \sigma_j \geq 0, j = 4, 5\}. \quad (3.170)$$

Let be defined in  $\Omega_\epsilon$  the coordinate transformation  $(\sigma_1, \sigma_2, \sigma_4, \sigma_5) \xrightarrow{\varphi=f(\sigma_1)} (\varphi, \sigma_2, \sigma_4, \sigma_5)$  where  $f(\sigma_1)$  tends to zero as  $\sigma_1$  tends to  $+\infty$  and has been chosen so that the conditions i)-iii) of definition 27 are satisfied with  $k = 2$ .

The flow of (3.134)-(3.138) defined on  $\Sigma_\epsilon$  is topologically equivalent (under  $f \circ \pi_1$ ) to the flow of the 4-dimensional dynamical system

$$\varphi' = \sqrt{\frac{2}{3}} \overline{f}' \sigma_2, \quad (3.171)$$

$$\begin{aligned} \sigma_2' &= \sigma_2^3 + \left( \frac{2\sigma_5^2}{3} - 1 \right) \sigma_2 - \frac{(\overline{W}_V + N) \sigma_4^2}{\sqrt{6}} + \\ &+ \left( \frac{(\overline{W}_\chi + M)(4 - 3\gamma)}{2\sqrt{6}} + \frac{\sigma_2 \gamma}{2} \right) (1 - \sigma_2^2 - \sigma_4^2 - \sigma_5^2) \end{aligned} \quad (3.172)$$

$$\sigma_4' = \frac{1}{6} \sigma_4 \left( \sqrt{6} (\overline{W}_V + N) \sigma_2 + 3(2 - \gamma) \sigma_2^2 + 3\gamma(1 - \sigma_4^2) + (4 - 3\gamma) \sigma_5^2 \right), \quad (3.173)$$

$$\sigma_5' = \frac{1}{6} \sigma_5 \left( 3(2 - \gamma) \sigma_2^2 - 3\gamma \sigma_4^2 - (4 - 3\gamma)(1 - \sigma_5^2) \right), \quad (3.174)$$

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<sup>11</sup> See 26 for the definition of functions with bar.

defined in the phase space <sup>12</sup>

$$\Omega_\epsilon = \{(\varphi, \sigma_2, \sigma_4, \sigma_5) \in \mathbb{R}^4 : 0 \leq \varphi \leq f(\epsilon^{-1}), \sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1, \sigma_4 \geq 0, \sigma_5 \geq 0\}. \quad (3.175)$$

It can be easily proved that (3.175) defines a manifold with boundary of dimension 4. Its boundary,  $\partial\Psi$ , is the union of the sets  $\{p \in \Omega_\epsilon : \varphi = 0\}$ ,  $\{p \in \Omega_\epsilon : \varphi = f(\epsilon^{-1})\}$ ,  $\{p \in \Omega_\epsilon : \sigma_4 = 0\}$ ,  $\{p \in \Omega_\epsilon : \sigma_5 = 0\}$  with the unitary 3-sphere.

### 3.5.6.1 Singular points of the flow of (3.171)-(3.174) in the phase space (3.175).

The system (3.171)-(3.174) admits the following singular points

1. The singular point  $P_1$  with coordinates  $\varphi = 0, \sigma_2 = -1, \sigma_4 = 0, \sigma_5 = 0$  always exists. The eigenvalues of the linearization around the singular point are  $0, \frac{1}{3}, 1 - \frac{N}{\sqrt{6}}, \frac{M(4-3\gamma)}{\sqrt{6}} - \gamma + 2$ . This means that the singular point is non-hyperbolic thus the Hartman-Grobmann theorem does not apply. However, by applying the Invariant Manifold theorem, we obtain that:

- (a)  $P_1$  has a 1-dimensional center manifold tangent to the  $\varphi$ -axis provided  $N \neq \sqrt{6}$  and  $M \neq -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$  (otherwise the center manifold would be 2- or 3-dimensional).
- (b)  $P_1$  admits a 3-dimensional unstable manifold and a 1-dimensional center manifold for
  - i.  $N < \sqrt{6}, 0 < \gamma < \frac{4}{3}, M > -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$ ; or
  - ii.  $N < \sqrt{6}, \frac{4}{3} < \gamma < 2, M < -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$ .

In this case the center manifold of  $P_1$  acts as a local source for an open set of orbits in (3.175).

- (c)  $P_1$  admits a 2-dimensional unstable manifold, a 1-dimensional stable manifold and a 1-dimensional center if
  - i.  $N > \sqrt{6}, 0 < \gamma < \frac{4}{3}, M > -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$ ; or
  - ii.  $N > \sqrt{6}, \frac{4}{3} < \gamma < 2, M < -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$ ; or
  - iii.  $N < \sqrt{6}, 0 < \gamma < \frac{4}{3}, M < -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$ ; or
  - iv.  $N < \sqrt{6}, \frac{4}{3} < \gamma < 2, M > -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$ .
- (d)  $P_1$  admits a 1-dimensional unstable manifold, a 2-dimensional stable manifold and a 1-dimensional center manifold for
  - i.  $N > \sqrt{6}, 0 < \gamma < \frac{4}{3}, M < -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$ ; or

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<sup>12</sup>For notational simplicity we will denote the image of  $\Omega_\epsilon$  under  $f$  by the same symbol.

$$\text{ii. } N > \sqrt{6}, \frac{4}{3} < \gamma < 2, M > -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}.$$

2. The singular point  $P_2$  with coordinates  $\varphi = 0, \sigma_2 = 1, \sigma_4 = 0, \sigma_5 = 0$  always exists. The eigenvalues of the linearization around the singular point are  $0, \frac{1}{3}, 1 + \frac{N}{\sqrt{6}}, -\gamma + \frac{M(3\gamma-4)}{\sqrt{6}} + 2$ . As before, let us determine conditions on the free parameters for the existence of center, unstable and stable manifolds for  $P_2$ .

(a) If  $N \neq -\sqrt{6}$  and  $M \neq \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$  there exists a 1-dimensional center manifold tangent to the  $\varphi$ -axis, otherwise the center manifold would be 2- or 3-dimensional.

(b)  $P_2$  has a 3-dimensional unstable manifold a 1-dimensional center manifold (tangent the  $\varphi$ -axis) if

$$\text{i. } N > -\sqrt{6}, 0 < \gamma < \frac{4}{3}, M < \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}; \text{ or}$$

$$\text{ii. } N > -\sqrt{6}, \frac{4}{3} < \gamma < 2, M > \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}.$$

In this case the center manifold of  $P_2$  acts as a local source for an open set of orbits in (3.175).

(c)  $P_2$  has a 2-dimensional unstable manifold a 1-dimensional stable and a 1-dimensional center manifold if

$$\text{i. } N < -\sqrt{6}, 0 < \gamma < \frac{4}{3}, M < \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}; \text{ or}$$

$$\text{ii. } N < -\sqrt{6}, \frac{4}{3} < \gamma < 2, M > \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}; \text{ or}$$

$$\text{iii. } N > -\sqrt{6}, 0 < \gamma < \frac{4}{3}, M > \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}; \text{ or}$$

$$\text{iv. } N > -\sqrt{6}, \frac{4}{3} < \gamma < 2, M < \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}.$$

(d)  $P_2$  has a 1-dimensional unstable manifold a 2-dimensional stable and a 1-dimensional center manifold if

$$\text{i. } N < -\sqrt{6}, 0 < \gamma < \frac{4}{3}, M > \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}; \text{ or}$$

$$\text{ii. } N < -\sqrt{6}, \frac{4}{3} < \gamma < 2, M < \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}.$$

3. The singular point  $P_3$  with coordinates  $\varphi = 0, \sigma_2 = \frac{M(3\gamma-4)}{\sqrt{6}(\gamma-2)}, \sigma_4 = 0, \sigma_5 = 0$  exists for

$$\text{(a) } 0 < \gamma < \frac{4}{3}, -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4} \leq M \leq \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}; \text{ or}$$

$$\text{(b) } \frac{4}{3} < \gamma < 2, \frac{\sqrt{6}(\gamma-2)}{3\gamma-4} \leq M \leq -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}.$$

The eigenvalues of the linearization are  $0, \lambda_1 = -\frac{(3\gamma-4)((3\gamma-4)M^2-2\gamma+4)}{12(\gamma-2)}, \lambda_2 = \frac{-M^2(4-3\gamma)^2+6(\gamma-2)\gamma+2MN(3\gamma-4)}{12(\gamma-2)}, \lambda_3 = \frac{6(\gamma-2)^2-M^2(4-3\gamma)^2}{12(\gamma-2)}$ . As before, let us determine conditions on the free parameters for the existence of center, unstable and stable manifolds for  $P_3$ .

- (a) For  $\gamma, N$  and  $M$  such that  $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$  the center manifold is 1-dimensional and tangent to the  $\varphi$ -axis. Otherwise the center manifold could be 2-, or 3-dimensional (it is never 4-dimensional).
- (b)  $P_3$  admits a 1-dimensional center manifold and a 3-dimensional stable manifold for
- $0 < \gamma < \frac{4}{3}, -\frac{\sqrt{2}\sqrt{\gamma-2}}{\sqrt{3\gamma-4}} < M < 0, N > \frac{M^2(4-3\gamma)^2-6(\gamma-2)\gamma}{2M(3\gamma-4)}$ ; or
  - $0 < \gamma < \frac{4}{3}, 0 < M < \frac{\sqrt{2}\sqrt{\gamma-2}}{\sqrt{3\gamma-4}}, N < \frac{M^2(4-3\gamma)^2-6(\gamma-2)\gamma}{2M(3\gamma-4)}$ .
- (c) In the cases
- $0 < \gamma < \frac{4}{3}, -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4} < M < -\frac{\sqrt{2}\sqrt{\gamma-2}}{\sqrt{3\gamma-4}}, N < \frac{M^2(4-3\gamma)^2-6(\gamma-2)\gamma}{2M(3\gamma-4)}$ ; or
  - $0 < \gamma < \frac{4}{3}, \frac{\sqrt{2}\sqrt{\gamma-2}}{\sqrt{3\gamma-4}} < M < \frac{\sqrt{6}(\gamma-2)}{3\gamma-4}, N > \frac{M^2(4-3\gamma)^2-6(\gamma-2)\gamma}{2M(3\gamma-4)}$ ; or
  - $\frac{4}{3} < \gamma < 2, \frac{\sqrt{6}(\gamma-2)}{3\gamma-4} < M < 0, N > \frac{M^2(4-3\gamma)^2-6(\gamma-2)\gamma}{2M(3\gamma-4)}$ ; or
  - $\frac{4}{3} < \gamma < 2, M = 0, N \in \mathbb{R}$ ; or
  - $\frac{4}{3} < \gamma < 2, 0 < M < -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}, N < \frac{M^2(4-3\gamma)^2-6(\gamma-2)\gamma}{2M(3\gamma-4)}$ , the unstable manifold is 2-dimensional (hence the stable manifold and the center manifold are both 1-dimensional).
  - Otherwise,  $P_3$  has a 1-dimensional unstable manifold. Thus, it is never a local source since its unstable manifold is of dimension less than 3.
4. The singular point  $R_1$  with coordinates  $\varphi = 0, \sigma_2 = 0, \sigma_4 = 0, \sigma_5 = 1$  always exists. The eigenvalues of the linearization are  $0, \frac{2}{3}, -\frac{1}{3}, \frac{4}{3} - \gamma$ . The center manifold is 1-dimensional and tangent to the  $\varphi$ -axis. The unstable (stable) manifold is 1-dimensional (2-dimensional) if  $\gamma > \frac{4}{3}$  otherwise it is 2-dimensional (1-dimensional).
5. The singular point  $R_2$  with coordinates  $\sigma_2 = \frac{\sqrt{\frac{2}{3}}}{M}, \sigma_4 = 0, \sigma_5 = \frac{\sqrt{\frac{4-2\gamma}{M^2}+3\gamma-4}}{\sqrt{3\gamma-4}}$  exists for  $0 < \gamma < \frac{4}{3}, M^2 \geq \frac{2(\gamma-2)}{3\gamma-4}$ . The eigenvalues of the linearization are  $0, -\frac{M+\sqrt{3M^2(4\gamma-5)-8(\gamma-2)}}{6M}, \frac{\sqrt{3M^2(4\gamma-5)-8(\gamma-2)}-M}{6M}, \frac{1}{3} \left( \frac{N}{M} + 2 \right)$ . Let us determine conditions on the free parameters for the existence of center, unstable and stable manifolds for  $R_2$ .
- (a)  $R_2$  has a 3-dimensional stable manifold and a 1-dimensional center manifold if
- $0 < \gamma < \frac{5}{4}, -2\sqrt{\frac{2}{3}}\sqrt{\frac{\gamma-2}{4\gamma-5}} \leq M < -\sqrt{2}\sqrt{\frac{\gamma-2}{3\gamma-4}}, N > -2M$ ; or
  - $0 < \gamma < \frac{5}{4}, \sqrt{2}\sqrt{\frac{\gamma-2}{3\gamma-4}} < M \leq 2\sqrt{\frac{2}{3}}\sqrt{\frac{\gamma-2}{4\gamma-5}}, N < -2M$ ; or
  - $\frac{5}{4} \leq \gamma < \frac{4}{3}, M < -\sqrt{2}\sqrt{\frac{\gamma-2}{3\gamma-4}}, N > -2M$ ; or
  - $\frac{5}{4} \leq \gamma < \frac{4}{3}, M > \sqrt{2}\sqrt{\frac{\gamma-2}{3\gamma-4}}, N < -2M$ ; or

- v.  $0 < \gamma < \frac{5}{4}, M < -2\sqrt{\frac{2}{3}}\sqrt{\frac{\gamma-2}{4\gamma-5}}, N > -2M$ ; or
- vi.  $0 < \gamma < \frac{5}{4}, M > 2\sqrt{\frac{2}{3}}\sqrt{\frac{\gamma-2}{4\gamma-5}}, N < -2M$ .

(b) By reversing the sign of the last inequality, i.e., the inequality solved for  $N$ , in the previous six cases we obtain conditions for  $R_2$  having a 2-dimensional stable manifold, a 1-dimensional unstable manifold and a 1-dimensional center manifold.

6. The singular point  $P_4$  with coordinates  $\varphi = 0, \sigma_2 = -\frac{N}{\sqrt{6}}, \sigma_4 = \sqrt{1 - \frac{N^2}{6}}, \sigma_5 = 0$  exists whenever  $N^2 < 6$ . The eigenvalues of the linearization are

$0, \frac{1}{6}(N^2 - 6), \frac{1}{6}(N^2 - 4), \frac{1}{3}N(2M + N) - \frac{1}{2}(MN + 2)\gamma$ . The conditions for the existence of stable, unstable and center manifolds is as follows.

(a) The center manifold is 1-dimensional and the stable manifold is 3-dimensional provided

- i.  $N = 0, M \in \mathbb{R}, \gamma \neq \frac{4}{3}$ ; or
- ii.  $0 < \gamma < \frac{4}{3}, -2 < N < 0, M > \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or
- iii.  $0 < \gamma < \frac{4}{3}, 0 < N < 2, M < \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or
- iv.  $\frac{4}{3} < \gamma < 2, -2 < N < 0, M < \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or
- v.  $\frac{4}{3} < \gamma < 2, 0 < N < 2, M > \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ .

(b) The stable manifold is 2-dimensional, the unstable manifold is 1-dimensional and the center manifold is 1-dimensional provided

- i.  $0 < \gamma < \frac{4}{3}, -\sqrt{6} < N < -2, M > \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or
- ii.  $0 < \gamma < \frac{4}{3}, -2 < N < 0, M < \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or
- iii.  $0 < \gamma < \frac{4}{3}, 0 < N < 2, M > \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or
- iv.  $0 < \gamma < \frac{4}{3}, 2 < N < \sqrt{6}, M < \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or
- v.  $\frac{4}{3} < \gamma < 2, -\sqrt{6} < N < -2, M < \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or
- vi.  $\frac{4}{3} < \gamma < 2, -2 < N < 0, M > \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or
- vii.  $\frac{4}{3} < \gamma < 2, 0 < N < 2, M < \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or
- viii.  $\frac{4}{3} < \gamma < 2, 2 < N < \sqrt{6}, M > \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ .

(c) The stable manifold is 1-dimensional, the unstable manifold is 2-dimensional and the center manifold is 1-dimensional provided

- i.  $0 < \gamma < \frac{4}{3}, -\sqrt{6} < N < -2, M < \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or
- ii.  $0 < \gamma < \frac{4}{3}, 2 < N < \sqrt{6}, M > \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or

- iii.  $\frac{4}{3} < \gamma < 2, -\sqrt{6} < N < -2, M > \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ ; or
- iv.  $\frac{4}{3} < \gamma < 2, 2 < N < \sqrt{6}, M < \frac{2(N^2-3\gamma)}{N(3\gamma-4)}$ .

7. The singular point  $R_3$  with coordinates  $\varphi = 0, \sigma_2 = -\frac{2\sqrt{\frac{2}{3}}}{N}, \sigma_4 = \frac{2}{\sqrt{3}|N|}, \sigma_5 = \frac{\sqrt{N^2-4}}{|N|}$  exists for  $N^2 \geq 4$ . The eigenvalues of the linearization are  $0, \frac{1}{6} \left( -\frac{\sqrt{64N^2-15N^4}}{N^2} - 1 \right), \frac{1}{6} \left( \frac{\sqrt{64N^2-15N^4}}{N^2} - 1 \right), -\frac{(2M+N)(3\gamma-4)}{3N}$ . The conditions for the existence of stable, unstable and center manifolds are as follows.

- (a) The stable manifold is 3-dimensional and the center manifold is 1-dimensional provided
  - i.  $0 < \gamma < \frac{4}{3}, N < -\frac{8}{\sqrt{15}}, M > -\frac{N}{2}$ ; or
  - ii.  $0 < \gamma < \frac{4}{3}, -\frac{8}{\sqrt{15}} \leq N < -2, M > -\frac{N}{2}$ ; or
  - iii.  $0 < \gamma < \frac{4}{3}, 2 < N \leq \frac{8}{\sqrt{15}}, M < -\frac{N}{2}$ ; or
  - iv.  $0 < \gamma < \frac{4}{3}, N > \frac{8}{\sqrt{15}}, M < -\frac{N}{2}$ ; or
  - v.  $\frac{4}{3} < \gamma < 2, N < -\frac{8}{\sqrt{15}}, M < -\frac{N}{2}$ ; or
  - vi.  $\frac{4}{3} < \gamma < 2, -\frac{8}{\sqrt{15}} \leq N < -2, M < -\frac{N}{2}$ ; or
  - vii.  $\frac{4}{3} < \gamma < 2, 2 < N \leq \frac{8}{\sqrt{15}}, M > -\frac{N}{2}$ ; or
  - viii.  $\frac{4}{3} < \gamma < 2, N > \frac{8}{\sqrt{15}}, M > -\frac{N}{2}$ .
- (b) By reversing the sign of the last inequality, i.e., the inequality solved for  $M$ , in the previous eight cases we obtain conditions for  $R_3$  having a 2-dimensional stable manifold, a 1-dimensional unstable manifold and a 1-dimensional center manifold.

8. The singular point  $P_5$  with coordinates

$\varphi = 0, \sigma_2 = \frac{\sqrt{6}\gamma}{M(3\gamma-4)-2N}, \sigma_4 = \frac{\sqrt{M^2(4-3\gamma)^2+MN(8-6\gamma)-6(\gamma-2)\gamma}}{2N+M(4-3\gamma)}, \sigma_5 = 0$  exists for  $2(2M+N) > 3M\gamma, M(3\gamma-4)(M(3\gamma-4)-2N) \geq 6(\gamma-2)\gamma$ , and  $\frac{3(MN+2)\gamma-2N(2M+N)}{(2N+M(4-3\gamma))^2} \leq 0$ . The eigenvalues of the linearization are  $0, \frac{12M+6N-3(3M+N)\gamma+\sqrt{3}\sqrt{f(\gamma,M,N)}}{6(M(3\gamma-4)-2N)}, \frac{3N(\gamma-2)+3M(3\gamma-4)+\sqrt{3}\sqrt{f(\gamma,M,N)}}{6(2N+M(4-3\gamma))}$ , and  $\frac{(2M+N)(3\gamma-4)}{6N+3M(4-3\gamma)}$ , where  $f(\gamma, M, N) = 2M^3N(3\gamma-4)^3 + 2MN(4N^2-6\gamma^2+3\gamma-6)(3\gamma-4) - M^2(8N^2-12\gamma-3)(4-3\gamma)^2 + 3(\gamma-2)(N^2(9\gamma-2)-24\gamma^2)$ . The stability conditions of  $P_5$  are very complicated to display them here. Thus we must rely on numerical experimentation. We can obtain, however, some analytic results. For instance, there exists at least a 1-dimensional center manifold. The unstable manifold is always of dimension lower than 3. Thus the singular point is never a local source. If all the eigenvalues, apart from the zero one, have negative real parts, then the center manifold of  $P_5$



acts as a local sink. This means that the orbits in the stable manifold approach the center manifold of  $P_5$  when the time goes forward.

9. The singular point  $P_6$  with coordinates

$\varphi = 0, \sigma_2 = \frac{\sqrt{6}\gamma}{M(3\gamma-4)-2N}, \sigma_4 = -\frac{\sqrt{M^2(4-3\gamma)^2+MN(8-6\gamma)-6(\gamma-2)\gamma}}{2N+M(4-3\gamma)}, \sigma_5 = 0$  exists for  $M(3\gamma-4)(M(3\gamma-4)-2N) \geq 6(\gamma-2)\gamma$ ,  $2(2M+N) < 3M\gamma$ , and  $\frac{3(MN+2)\gamma-2N(2M+N)}{(2N+M(4-3\gamma))^2} \leq 0$ . The eigenvalues of the linearization are the same displayed in the previous point. However the stability conditions are rather different (since the existence conditions are different from those of  $P_5$ ). As before, the stability conditions are very complicated to display them here, but similar conclusions concerning the center and unstable manifold, as for  $P_5$ , are obtained. For get further information about its stability we must to resort to numerical experimentation.

### 3.5.6.2 Physical description of the solutions and connection with observables.

Let us now present the formalism of obtaining the physical description of a singular point, and also connect with the basic observables relevant for a physical discussion. These will allow us to describe the cosmological behavior of each singular point, in the next section.

Firstly, around a singular point we obtain first-order expansions for  $H$ ,  $a$ ,  $\phi$ , and  $\rho$  and  $\rho_r$  in terms of  $t$ , considering equations: (3.2); the definition of the scale factor  $a$  in terms of the Hubble factor  $H$ ; the definition of  $\sigma_2$ ; the matter conservation equations (3.3) and (3.4), respectively, given by

$$\begin{aligned} 2\dot{H}(t) &= H(t)^2 (3(\gamma-2)\sigma_2^{*2} + 3\gamma(\sigma_4^{*2} + \sigma_5^{*2} - 1) - 4\sigma_5^{*2}), \\ \dot{a}(t) &= a(t)H(t), \\ \dot{\phi}(t) &= \sqrt{6}\sigma_2^*H(t), \\ \dot{\rho}(t) &= -\frac{3}{2}H(t)^3 \left( \sqrt{6}M(3\gamma-4)\sigma_2^* - 6\gamma \right) (\sigma_2^{*2} + \sigma_4^{*2} + \sigma_5^{*2} - 1), \\ \dot{\rho}_r(t) &= -12\sigma_5^{*2}H(t)^3, \end{aligned} \quad (3.176)$$

where the star-superscript denotes the evaluation at a specific singular point. The equation

$$\ddot{\phi}(t) = \frac{3}{2}H(t)^2 \left( M(3\gamma-4)(\sigma_2^{*2} + \sigma_4^{*2} + \sigma_5^{*2} - 1) - 2(N\sigma_4^{*2} + \sqrt{6}\sigma_2^*) \right), \quad (3.177)$$

derived from the equation of motion for the scalar field (3.5) should be used as a consistency test for the above procedure. Solving the differential equations (3.176) and substituting the resulting expressions in the equation (3.177) results in

$$\begin{aligned} &-6M(3\gamma-4)(\sigma_2^{*2} + \sigma_4^{*2} + \sigma_5^{*2} - 1) + 12N\sigma_4^{*2} + \\ &+ 2\sqrt{6}\sigma_2^*(3\gamma(\sigma_2^{*2} + \sigma_4^{*2} + \sigma_5^{*2} - 1) - 6\sigma_2^{*2} - 4\sigma_5^{*2} + 6) = 0. \end{aligned} \quad (3.178)$$

This integrability condition should be (at least asymptotically) fulfilled.

Table 3.5: Observable cosmological quantities, and physical behavior of the solutions, at the singular points of the cosmological system. We use the notations  $M_1(\gamma) = \frac{\sqrt{2\gamma(3\gamma-8)+8}}{4-3\gamma}$ ,  $M_2(\gamma) = \frac{\sqrt{6}\sqrt{(\gamma-3)\gamma+2}}{4-3\gamma}$

Cr.P.	$q$	$w_{\text{eff}}$	Solution/description
$P_1$	2	1	Decelerating.
$P_2$	2	1	Decelerating.
$P_3$	$\frac{-M^2(4-3\gamma)^2+2\gamma(3\gamma-8)+8}{4(\gamma-2)}$	$-\frac{M^2(4-3\gamma)^2}{6(\gamma-2)} + \gamma - 1$	Accelerating for $0 < \gamma < \frac{2}{3}$ $-M_1(\gamma) < M < M_1(\gamma)$
$P_4$	$\frac{1}{2}(N^2 - 2)$	$\frac{1}{3}(N^2 - 3)$	Accelerating for $-\sqrt{2} < n < \sqrt{2}$ powerlaw-inflationary
$P_5$	$\frac{3(M+N)\gamma-2(2M+N)}{2N+M(4-3\gamma)}$	$\frac{M(4-3\gamma)-2N(\gamma-1)}{M(3\gamma-4)-2N}$	Accelerating for $\frac{3(M+N)\gamma-2(2M+N)}{2N+M(4-3\gamma)} < 0$ matter-kinetic-potential- scaling.
$P_6$	$\frac{3(M+N)\gamma-2(2M+N)}{2N+M(4-3\gamma)}$	$\frac{M(4-3\gamma)-2N(\gamma-1)}{M(3\gamma-4)-2N}$	Accelerating for $\frac{3(M+N)\gamma-2(2M+N)}{2N+M(4-3\gamma)} < 0$ matter-kinetic-potential- scaling.
$R_1$	1	$\frac{1}{3}$	Decelerating. Radiation-dominated.
$R_2$	1	$\frac{1}{3}$	Decelerating. radiation-kinetic-potential- scaling.
$R_3$	1	$\frac{1}{3}$	Decelerating. radiation-kinetic-potential- scaling.

Instead of apply this procedure to a generic singular point here, we submit the reader to section 3.6 for some worked examples where this procedure has been applied. However we will discuss on some cosmological observables.

We can calculate the deceleration parameter  $q$  defined as usual as [1]

$$q = -\frac{a\ddot{a}}{a^2}. \quad (3.179)$$

Additionally, we can calculate the effective (total) equation-of-state parameter of the universe  $w_{\text{eff}}$ , defined conventionally as

$$w_{\text{eff}} \equiv \frac{p_{\text{tot}}}{\rho_{\text{tot}}}, \quad (3.180)$$

where  $p_{\text{tot}}$  and  $\rho_{\text{tot}}$  are respectively the total isotropic pressure and the total energy density. Therefore, in terms of the auxiliary variables we have

$$q = -\frac{3}{2}(\gamma - 2)\sigma_2^2 - \frac{3\gamma\sigma_4^2}{2} + \frac{1}{2}(4 - 3\gamma)\sigma_5^2 + \frac{1}{2}(3\gamma - 2) \quad (3.181)$$

$$w_{\text{eff}} = (2 - \gamma)\sigma_2^2 - \gamma\sigma_4^2 + \frac{1}{3}(4 - 3\gamma)\sigma_5^2 + \gamma - 1. \quad (3.182)$$

First of all, for each singular point described in the last section we calculate the effective (total) equation-of-state parameter of the universe  $w_{\text{eff}}$  using (3.182), and the deceleration parameter  $q$  using (3.181). The results are presented in Table 3.5. Furthermore, as usual, for an expanding universe  $q < 0$  corresponds to accelerating expansion and  $q > 0$  to decelerating expansion.

### 3.5.7 The Flow as $\phi \rightarrow -\infty$

With the purpose of complementing the global analysis of the system it is necessary investigate its behavior as  $\phi \rightarrow -\infty$ . It is an easy task since the system (3.134)-(3.138) is invariant under the transformation of coordinates

$$(\phi, \sigma_2) \rightarrow -(\phi, \sigma_2), \quad V \rightarrow U, \quad \chi \rightarrow \Xi,$$

where  $U(\phi) = V(-\phi)$  and  $\Xi(\phi) = \chi(-\phi)$ . Hence, for a particular potential  $V$ , and a particular coupling function  $\chi$ , the behavior of the solutions of the equations (3.134)-(3.138) around  $\phi = -\infty$  is equivalent (except for the sign of  $\phi$ ) to the behavior of the system near  $\phi = \infty$  with potential and coupling functions  $U$  and  $\Xi$ , respectively.

If  $U$  and  $\Xi$  are of class  $\mathcal{E}_+^2$ , the preceding analysis in  $\bar{\Sigma}_\epsilon$  can be applied (with and adequate choice of  $\epsilon$ ).

The set of class  $C^k$  functions well behaved in both  $+\infty$  and  $-\infty$  is denoted by  $\mathcal{E}^k$ . Latin uppercase letters with subscripts  $+\infty$  and  $-\infty$ , are used respectively to indicate the exponential order of  $\mathcal{E}^k$  functions in  $+\infty$  and in  $-\infty$ .

## 3.6 Examples

In this section we apply the mathematics discussed in previous sections to several worked examples from both analytical and numerical viewpoint.

### 3.6.1 Numerical Evidence of the Result of Theorem 26

For the particular case  $\chi(\sigma_1) = e^{M\sigma_1}$ , from equation (3.168), follows that  $\sigma_2 = \sigma_{2c} := \frac{M(4-3\gamma)}{\sqrt{6}(2-\gamma)}$  is an invariant set. Given  $\frac{4}{3} < \gamma < 2$ , the existence conditions lead to  $M_1(\gamma) \leq M \leq -\frac{\sqrt{6}(\gamma-2)}{3\gamma-4}$ , where  $M_1(\gamma) = \frac{\sqrt{2\gamma(3\gamma-8)+8}}{4-3\gamma}$ . For such values  $\frac{\partial f_2}{\partial \sigma_2}|_{\sigma_{2c}} = \frac{M^2(4-3\gamma)^2-6(\gamma-2)^2}{12(\gamma-2)} \geq 0$ . Thus the asymptotic phase configuration  $\sigma_1 \rightarrow -\text{sgn}\sigma_{2c}\infty, \sigma_2 \rightarrow \sigma_{2c}$  is never approached (for an open set of orbits) as  $\tau \rightarrow \infty$ . For the original system (i.e., taking the time reversal transformation) this means that this asymptotic phase configuration is never approached towards the past. In figure 3.3 we show the qualitative dynamics of the flow of (3.162) for the choice  $M = \sqrt{2/3}$ , and  $\gamma = 1$ . In order to compactify the phase space we have introduced the coordinate transformation  $\sigma_1 \rightarrow \tanh \sigma_1$ . The mentioned asymptotic configuration is represented in figure 3.3 by  $P_3^\pm$ .

### 3.6.2 Coupling Functions and Potentials of Exponential Orders $M = 0$ and $N = -\mu \neq 0$ , Respectively

As an example let us consider  $\chi, V \in \mathcal{E}_+^2$  of exponential orders  $M = 0$  and  $N = -\mu$ , respectively. This class of potentials contains the cases investigated in [409, 410] (there are not considered coupling to matter, i.e.,  $\chi(\phi) \equiv 1$ , in the second case, for flat FRW cosmologies), the case investigated in [415] (for positive potentials and standard FRW dynamics), the example examined in [136], etc. In table 3.6 are summarized the location, existence conditions and stability of the singular points.<sup>13</sup>

Let us discuss some physical properties of the cosmological solutions associated to the singular points displayed in table 3.2.

- $P_{1,2}$  represent kinetic-dominated cosmological solutions. They behave as stiff-like matter. The associated cosmological solution satisfies  $H = \frac{1}{3t-c_1}, a = \sqrt[3]{3t-c_1}c_2, \phi = c_3 \pm \sqrt{\frac{2}{3}} \ln(3t-c_1)$ , where  $c_j, j = 1, 2, 3$  are integration constants. These solutions are associated with the local past attractors of the systems for an open set of values of the parameter  $\mu$ .
- $P_3$  represents matter-dominated cosmological solutions that satisfy  $H = \frac{2}{3t\gamma-2c_1}, a = (3t\gamma-2c_1)^{\frac{2}{3\gamma}}c_2, \rho = \frac{12}{(3t\gamma-2c_1)^2} + c_3$ .

<sup>13</sup>The stability is analyzed for the flow restricted to the invariant set  $\varphi = 0$ , i.e., we are not taking into account perturbations in the  $\varphi$ -axis.

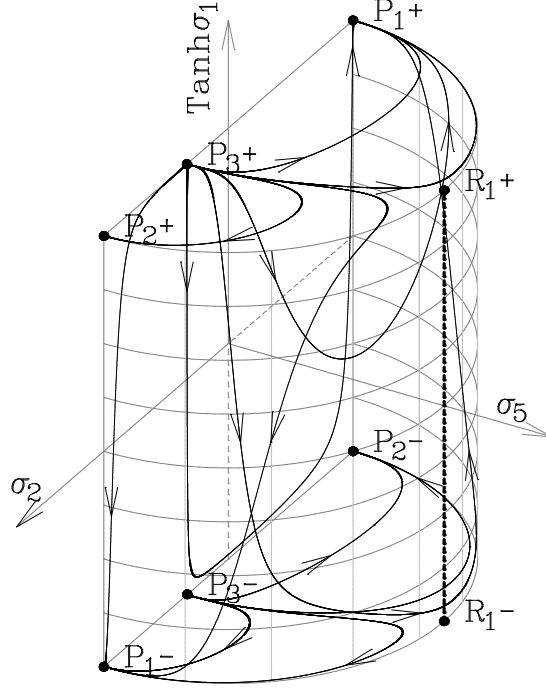


Figure 3.3: Qualitative dynamics of the flow of (3.162) for the choice  $M = \sqrt{2/3}$  and  $\gamma = 1$ . Observe that  $P_1^+$  or  $P_1^-$  are the local attractors (sinks).  $P_3^+$  is a local source,  $P_3^-$  is a saddle for the full dynamics, but it is a local source in the invariant set  $\tanh \sigma_1 = -1$ . The thick dashed line is an invariant set which is unstable. In fact all its points including  $R_1^+$  and  $R_1^-$ , act as saddle points. They correspond to cosmological radiation-dominated solutions.

- $R_1$  represents a radiation-dominated cosmological solutions satisfying  $H = \frac{1}{2t-c_1}$ ,  $a = \sqrt{2t-c_1}c_2$ ,  $\rho_r = \frac{3}{(2t-c_1)^2} + c_3$ .
- $P_4$  represents power-law scalar-field dominated inflationary cosmological solutions. As  $t \rightarrow 0^+$  the potential behaves as  $V \sim V_0 \exp[-\mu\phi]$ . Thus it is easy to obtain the asymptotic exact solution:  $H = \frac{2}{t\mu^2-2c_1}$ ,  $a = (t\mu^2-2c_1)^{\frac{2}{\mu^2}}c_2$ ,  $\phi \sim \frac{1}{\mu} \ln \left[ \frac{V_0(t\mu^2-2c_1)^2}{2(6-\mu^2)} \right]$ .
- $P_{5,6}$  represent matter-kinetic-potential scaling solutions. As before, in the limit  $t \rightarrow 0^+$  we obtain the asymptotic expansions:  $H = \frac{2}{3t\gamma-2c_1}$ ,  $a = (3t\gamma-2c_1)^{\frac{2}{3\gamma}}c_2$ ,  $\phi \sim \frac{1}{\mu} \ln \left[ \frac{V_0\mu^2(3t\gamma-2c_1)^2}{18(2-\gamma)\gamma} \right]$ .
- $R_3$  represent radiation-kinetic-potential scaling solutions. As before are deduced the following asymptotic expansions:  $H = \frac{1}{2t-c_1}$ ,  $a = \sqrt{2t-c_1}c_2$ ,  $\phi \sim \frac{1}{\mu} \ln \left[ \frac{v_0\mu^2(2t-c_1)^2}{4} \right]$ .

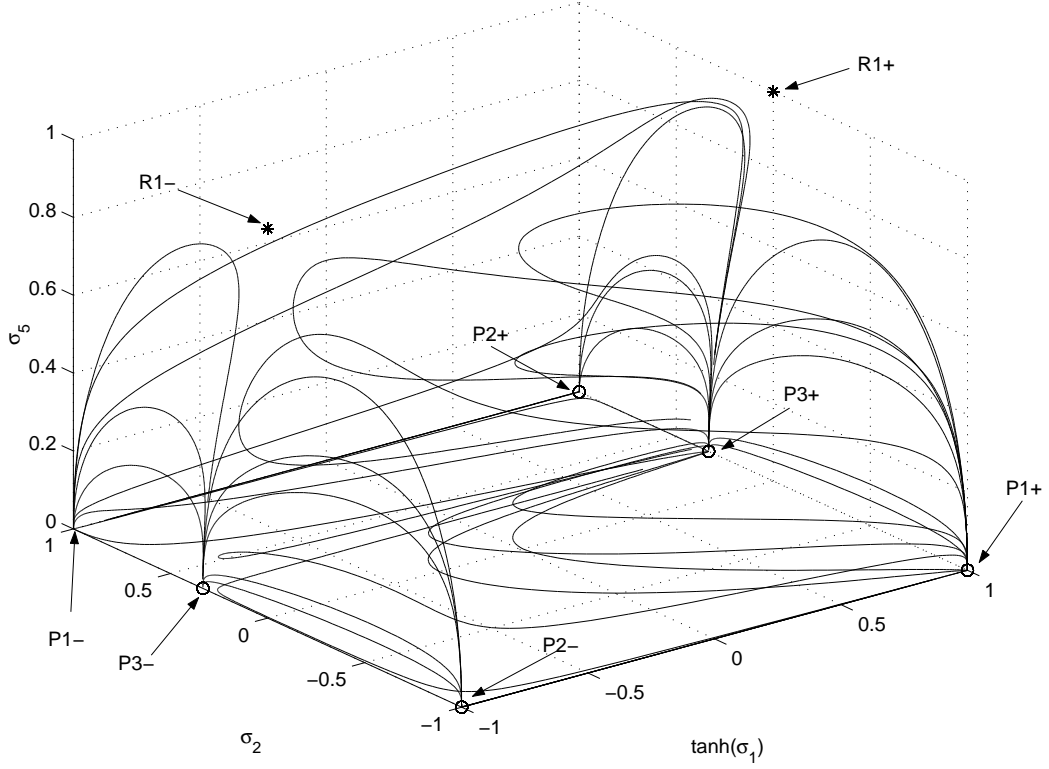


Figure 3.4: The graphic illustrates the result of theorem 26. We set  $M = \sqrt{2/3}$  and  $\gamma = 1$ . The point with  $\tanh \sigma_1 = \pm 1$  are the local sinks (thus for the original system (3.134)-(3.138) the scalar field almost always diverges towards the past).

### 3.6.2.1 Powerlaw coupling and Albrecht-Skordis potential in the invariant set $\rho_r = 0$

Let us consider the coupling function

$$\chi(\phi) = \left(\frac{3\alpha}{8}\right)^{\frac{1}{\alpha}} \chi_0 (\phi - \phi_0)^{\frac{2}{\alpha}}, \quad \alpha > 0, \text{const.}, \phi_0 \geq 0. \quad (3.183)$$

Observe that

$$\frac{d \ln \chi(\phi)}{d\chi} = \frac{2}{\alpha(\phi - \phi_0)} \neq 0$$

for all finite value of  $\phi$ . Since  $\ln \chi(\phi)$  has not stationary points thus the early time dynamics is associated to the limit where the scalar field diverges.

This choice produces a coupling BD parameter given by

$$2\omega(\chi) + 3 = \frac{4}{3}\alpha \left(\frac{\chi}{\chi_0}\right)^\alpha.$$

This types of power law couplings were investigated in [416] from the astrophysical viewpoint. For STTs without potential, the cosmological solutions for the matter domination era (in a Robertson-Walker metric) are  $a(t) \propto (\ln t)^{(\alpha-1)/3\alpha} t^{\frac{2}{3}}$ ,  $\phi(t) \propto (\ln t)^{\frac{1}{\alpha}}$ . The

Table 3.6: Location of the singular points of the flow of (3.171)-(3.174) defined in the invariant set  $\{p \in \Omega_\epsilon : \varphi = 0\}$  for  $M = 0$  and  $N = -\mu$ .

Label	$(\sigma_2, \sigma_4, \sigma_5)$	Existence	Stability <sup>a</sup>
$P_1$	$(-1, 0, 0)$	always	unstable if $\mu > -\sqrt{6}$
$P_2$	$(1, 0, 0)$	always	unstable if $\mu < \sqrt{6}$
$P_3$	$(0, 0, 0)$	always	saddle
$R_1$	$(0, 0, 1)$	always	saddle
$P_4$	$\left(\frac{\mu}{\sqrt{6}}, \sqrt{1 - \frac{\mu^2}{6}}, 0\right)$	$\mu^2 < 6$	stable for $\begin{cases} 0 < \gamma < \frac{4}{3}, \mu^2 < 3\gamma, \text{ or} \\ \frac{4}{3} < \gamma < 2, \mu^2 < 2 \end{cases},$ saddle otherwise
$P_{5,6}$	$\left(\sqrt{\frac{3}{2}\gamma}, \pm \frac{1}{\mu} \sqrt{\frac{3}{2}(2-\gamma)\gamma}, 0\right)$	$\mu^2 > 3\gamma$	stable for $\begin{cases} 0 < \gamma < \frac{2}{9}, \mu^2 > 3\gamma, \text{ or} \\ \frac{2}{9} < \gamma < \frac{4}{3}, 3\gamma < \mu^2 < \frac{24\gamma^2}{9\gamma-2}, \text{ or} \\ \frac{2}{9} < \gamma < \frac{4}{3}, \mu^2 > \frac{24\gamma^2}{9\gamma-2} \end{cases},$ saddle otherwise
$R_3$	$\left(\frac{2\sqrt{\frac{2}{3}}}{\mu}, \frac{2}{\sqrt{3} \mu }, \frac{\sqrt{\mu^2-4}}{ \mu }\right)$	$ \mu  > 2$	stable if $\frac{4}{3} < \gamma < 2$ , saddle otherwise

The stability is analyzed for the flow restricted to the invariant set  $\varphi = 0$ .

values of the parameter  $\alpha$  in concordance with the predictions of  ${}^4H$  are  $\alpha = 1, 0.33, 3$  (see table 4.2 in [416]). Let us consider also the Albrecht-Skordis potential given by (3.66).

Observe first that

$$W_\chi(\phi) = \chi'(\phi)/\chi(\phi) = \frac{2}{\alpha(\phi - \phi_0)} \Rightarrow \lim_{\phi \rightarrow +\infty} W_\chi(\phi) = 0 \quad (3.184)$$

and

$$W_V(\phi) = V'(\phi)/V(\phi) + \mu = \frac{2(\phi - B)}{A + (B - \phi)^2} \Rightarrow \lim_{\phi \rightarrow +\infty} W_V(\phi) = 0. \quad (3.185)$$

In other words, the coupling function (3.183) and the potential (3.66) are WBI of exponential orders  $M = 0$  and  $N = -\mu$ , respectively.

It is easy to prove that Power-law coupling and the Albrecht-Skordis potential are at

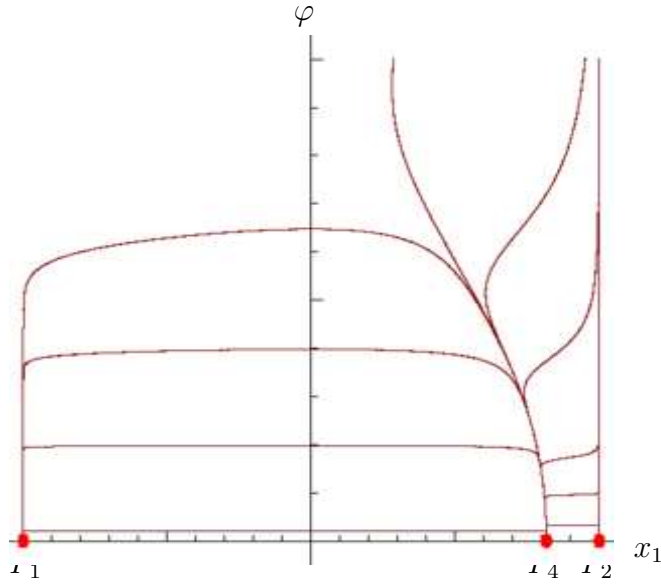


Figure 3.5: Orbits in the invariant set  $\{x_2 = 0\} \subset \bar{\Sigma}_\epsilon$  for the model with coupling function (3.183) potential (3.66). We select the values of the parameters:  $\epsilon = 1.00$ ,  $\mu = 2.00$ ,  $A = 0.50$ ,  $\alpha = 0.33$ ,  $B = 0.5$ ,  $\phi_0 = 0$ , and  $\gamma = 1$ . Observe that i) almost all the orbits are past asymptotic to  $P_1$ ; ii)  $P_2$  is a saddle, and iii) the center manifold of  $P_4$  attracts all the orbits in the  $\{x_2 = 0\}$ . However, it is no more an attractor in the invariant set  $x_2 > 0$ ,  $\varphi = 0$  (see figure 3.6)

least  $\mathcal{E}_+^2$ , under the admissible coordinate transformation <sup>14</sup>

$$\varphi = \phi^{-\frac{1}{2}} = f(\phi). \quad (3.186)$$

Using the above coordinate transformation we find

$$\overline{W}_\chi(\varphi) = \begin{cases} \frac{2\varphi^2}{\alpha(1-\varphi^2\phi_0)} & , \quad \varphi > 0 \\ 0 & , \quad \varphi = 0 \end{cases} \quad (3.187)$$

$$\overline{W}_V(\varphi) = \begin{cases} -\frac{2\varphi^2(B\varphi^2-1)}{A\varphi^4+(B\varphi^2-1)^2} & , \quad \varphi > 0 \\ 0 & , \quad \varphi = 0 \end{cases} \quad (3.188)$$

and

$$\overline{f}'(\varphi) = \begin{cases} -\frac{1}{2}\varphi^3 & , \quad \varphi > 0 \\ 0 & , \quad \varphi = 0 \end{cases} \quad (3.189)$$

In this example, the evolution equations for  $\varphi$ ,  $y$ , and  $z$  are given by the equations (3.78)-(3.80) with  $M = 0$ ,  $N = -\mu$  and  $\overline{W}_V$ ,  $\overline{f}'$ ,  $\overline{W}_\chi$  given respectively by (3.188), (3.189) and (3.187). The state space is defined by  $\bar{\Sigma}_\epsilon = \{(\varphi, x_1, x_2) | 0 \leq \varphi \leq \sqrt{\epsilon}, 0 \leq x_1^2 + x_2^2 \leq 1\}$ .

<sup>14</sup>We fix here an error in formulas B6-B9 in [136]. With the choice  $\varphi = \phi^{-1}$  the resulting barred functions given by B7-B9 there, are not of the desired differentiable class.



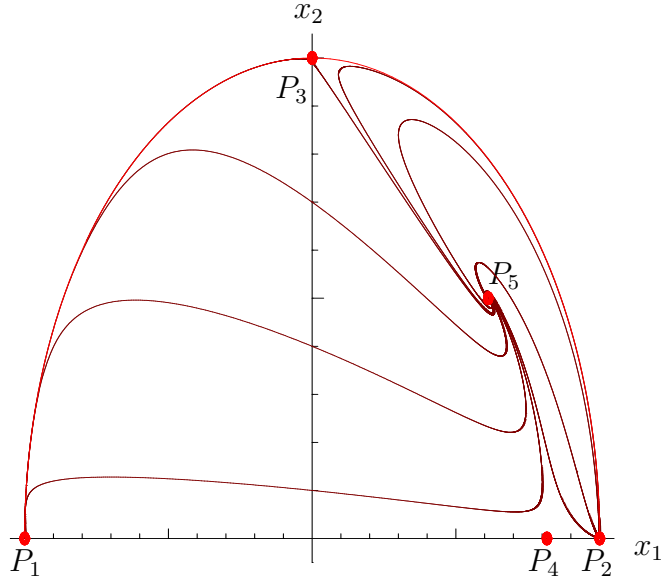


Figure 3.6: Orbits in the invariant set  $\{\varphi = 0\} \subset \bar{\Sigma}_\epsilon$  for the model with coupling function (3.183) and potential (3.66). We select the values of the parameters:  $\epsilon = 1.00$ ,  $\mu = 2.00$ ,  $A = 0.50$ ,  $\alpha = 0.33$ ,  $B = 0.5$ ,  $\phi_0 = 0$ , and  $\gamma = 1$ . In the figure i)  $P_{1,2}$  are local past attractors, but  $P_1$  is the global past attractor; ii)  $P_{3,4}$  are saddles, and iii)  $P_5$  is a local future attractor.

The singular points of the system (3.79-3.78) in this example are  $P_{1,2} = (0, \mp 1, 0)$ ,  $P_3 = (0, 0, 1)$ ,  $P_4 = \left(0, \frac{\mu}{\sqrt{6}}, 0\right)$ , and  $P_{5,6} = \left(0, \sqrt{\frac{3}{2}} \frac{\gamma}{\mu}, \mp \frac{\sqrt{-12\gamma + 4\mu^2}}{2\mu}\right)$ . The points  $P_{1,2,3}$  exist for all the values of the free parameters. The singular point  $P_4$  exists for  $\mu^2 \leq 6$ . The singular point  $P_5$  exists if  $\mu \leq -\sqrt{3\gamma}$  whereas the singular point  $P_6$  exists if  $\mu \geq \sqrt{3\gamma}$ . We will characterize the singular points  $P_{5,6}$  in more detail (for the analysis of the other singular points we submit the reader to table 3.2). The singular points  $P_{5,6}$  corresponds to those studied in the book [2] (see equation 4.23 p 49) with the identifications  $\Psi = x_1$ ,  $\Phi^2 = \frac{V(\phi)}{3H^2} = \frac{3\gamma(2-\gamma)}{2\mu^2}$  and  $k = -\mu$ . As stated in that reference the scalar field 'inherits' the equation of state of the fluid, i.e.,  $\gamma_\phi = \gamma$ . Then this solutions represents cosmological kinetic-matter scaling solutions<sup>15</sup> (the potential energy density is negligible). Because the scalar field mimics the perfect fluid with exact the same EoS at these points it seems reasonably to think that if one combine the two 'fluids' via  $p_{\text{tot}} = p_\phi + p$  and  $\rho_{\text{tot}}$  those singular points will corresponds to exact perfect fluid models with total EoS parameter equal to  $\gamma - 1$  (see [2] p 54). This is the case but the effective EoS parameter of total matter would be  $\omega_{\text{tot}} = \gamma \left(1 - \frac{3}{\mu^2}\right)$  instead  $\gamma - 1$ . This fact is due to the existence of the coupling. These singular points represent accelerating cosmologies for  $0 < \gamma < \frac{2}{3}$ . The eigenvalues of the matrix of derivatives evaluated at  $P_{5,6}$  are  $\left(0, -\frac{2-\gamma}{4\mu} \pm \frac{1}{4\mu} \sqrt{(2-\gamma)(24\gamma^2 + \mu^2(2-9\gamma))}\right)$ . The orbits initially in the stable sub-

<sup>15</sup>See reference [409, 417] for a notion of 'scaling' solutions, particularly, kinetic-matter scaling solutions.

space of  $P_{5,6}$  spiral-in around  $P_{5,6}$  if  $\mu^2 > 24\gamma^2/(-2 + 9\gamma)$  provided  $\frac{2}{9} < \gamma < 2, \gamma \neq \frac{4}{3}$ . Otherwise  $P_{5,6}$ , looks like an stable node for the orbits lying in the stable subspace. The center subspace is tangent to the singular points in the direction of the  $\varphi$  axis.

The system (3.79-3.78) can admit also the singular points

$$P_{7,8} = \left( \sqrt{\frac{\mu}{(B\mu + 1) \pm \sqrt{1 - A\mu^2}}}, 0, 0 \right).$$

$P_7$  exists for  $\mu < 0, A \leq \frac{1}{\mu^2}, B > \frac{\mu - \epsilon}{\epsilon\mu} + \sqrt{\frac{1 - A\mu^2}{\mu^2}}$  or  $\mu > 0, A \leq \frac{1}{\mu^2}, B > \frac{\mu - \epsilon}{\epsilon\mu} - \sqrt{\frac{1 - A\mu^2}{\mu^2}}$ , whereas,  $P_8$  exists for  $\mu < 0, A \leq \frac{1}{\mu^2}, B > \frac{\mu - \epsilon}{\epsilon\mu} - \sqrt{\frac{1 - A\mu^2}{\mu^2}}$ , or  $\mu > 0, A < \frac{1}{\mu^2}, B > \frac{\mu - \epsilon}{\epsilon\mu} + \sqrt{\frac{1 - A\mu^2}{\mu^2}}$ , or  $\mu > 0, A = \frac{1}{\mu^2}, B > \frac{\mu - \epsilon}{\epsilon\mu} - \sqrt{\frac{1 - A\mu^2}{\mu^2}}$ .

The eigenvalues of the linearization at  $P_7$  are  $-\frac{\gamma}{2}, -\frac{1}{2} - \frac{\sqrt{9A^2 + 12(A\mu^2 + \sqrt{1 - A\mu^2} - 1)A}}{6A}, -\frac{1}{2} + \frac{\sqrt{9A^2 + 12(A\mu^2 + \sqrt{1 - A\mu^2} - 1)A}}{6A}$ . From the existence conditions follows that it is a saddle point.

The eigenvalues of the linearization at  $P_8$  are  $-\frac{\gamma}{2}, -\frac{1}{2} - \frac{\sqrt{9A^2 - 12A(-A\mu^2 + \sqrt{1 - A\mu^2} + 1)}}{6A}, -\frac{1}{2} + \frac{\sqrt{9A^2 - 12A(-A\mu^2 + \sqrt{1 - A\mu^2} + 1)}}{6A}$ . Thus,  $P_8$  is an attractor for

- i)  $0 < A < \frac{8(2\mu^2 + 3)}{(4\mu^2 + 3)^2}, B > \frac{1}{\epsilon} - \frac{1}{\mu} + \sqrt{\frac{1}{\mu^2} - A}, \mu > 0$  or  $0 < A < \frac{8(2\mu^2 + 3)}{(4\mu^2 + 3)^2}, B > \frac{1}{\epsilon} - \frac{1}{\mu} - \sqrt{\frac{1}{\mu^2} - A}, \mu < 0$  (two complex eigenvalues with negative real part and one negative real eigenvalue) or
- ii)  $\frac{8(2\mu^2 + 3)}{(4\mu^2 + 3)^2} \leq A < \frac{1}{\mu^2}, B > \frac{1}{\epsilon} - \frac{1}{\mu} + \sqrt{\frac{1}{\mu^2} - A}, \mu > 0$  or  $\frac{8(2\mu^2 + 3)}{(4\mu^2 + 3)^2} \leq A < \frac{1}{\mu^2}, B > \frac{1}{\epsilon} - \frac{1}{\mu} + \sqrt{\frac{1}{\mu^2} - A}, \mu < 0$  (three real negative eigenvalues).

To finish this section let us re-examine the example discussed in [136] section B.1 in presence of radiation.

### 3.6.2.2 Powerlaw coupling and Albrecht-Skordis potential for the general model including radiation

As we investigated in section 3.5.4, the late time dynamics of the flow of (3.144)-(3.147) is associated with the extremes of the potential (the singular point  $P_2 = (0, 0, 0)$ ). When we restrict ourselves to this invariant set, we find that the singular point associated to  $\phi^+$  is always a saddle point of the corresponding phase portrait. The singular point associated to  $\phi^-$  could be either a stable node or a stable spiral if

$$\frac{8(3 + 2\mu^2)}{(3 + 4\mu^2)^2} < A \leq \frac{1}{\mu^2}$$

or

$$A < \frac{8(3 + 2\mu^2)}{(3 + 4\mu^2)^2}.$$

The early time dynamics of the flow of (3.144)-(3.147) corresponds to the limit  $\phi = +\infty$ .

In this example, the evolution equations for  $\varphi$ ,  $\sigma_2$ ,  $\sigma_4$ , and  $\sigma_5$  are given by the equations (3.171)-(3.174) with  $M = 0$ ,  $N = -\mu$ , and  $\overline{W}_\chi(\varphi)$ ,  $\overline{W}_V(\varphi) = 0$ , and  $\overline{f}'$ , given by (3.187), (3.188) and (3.189) respectively. The state space is defined by

$$\Omega_\epsilon = \{(\varphi, \sigma_2, \sigma_4, \sigma_5) \in \mathbb{R}^4 : 0 \leq \varphi \leq \sqrt{\epsilon}, \sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1, \sigma_4 \geq 0, \sigma_5 \geq 0\}.$$

In figure 3.7 we show some orbits in the invariant set  $\sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1$  for  $\varphi = 0$  for the model with coupling function (3.183) potential (3.66). We select the values of the parameters:  $\epsilon = 1.00$ ,  $\mu = 2.00$ ,  $A = 0.50$ ,  $\alpha = 0.33$ ,  $B = 0.5$ , and  $\phi_0 = 0$ . In this case  $P_5$  is the local sink in this invariant set. For this choice of parameters the points  $P_{7,8}$  do not exist. In the figure 3.8 are displayed some orbits in the invariant set  $\sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1$  for the choice of  $\varphi = 0$  for the model with coupling function (3.183) potential (3.66). We select the values of the parameters:  $\gamma = 1$ ,  $\epsilon = 1.00$ ,  $\mu = 2.10$ ,  $A = 0.50$ ,  $\alpha = 0.33$ ,  $B = 0.5$ , and  $\phi_0 = 0$ . The dynamics is essentially the same as in the figure 3.7 with the difference that in this case  $R_3$  exists and it is a saddle.

### 3.6.3 Quadratic Gravity: $F(R) = R + \alpha R^2$ .

Table 3.7: Location of the singular points of the flow of (3.171)-(3.174) defined in the invariant set  $\{p \in \Omega_\epsilon : \varphi = 0\}$  for  $M = \sqrt{2/3}$  and  $N = 0$ .

Label	$(\sigma_2, \sigma_4, \sigma_5)$	Existence	Stability <sup>a</sup>
$P_1$	$(-1, 0, 0)$	always	unstable for $\begin{cases} 0 < \gamma < \frac{4}{3}, \text{ or} \\ \frac{4}{3} < \gamma < \frac{5}{3} \end{cases}$ saddle, otherwise
$P_2$	$(1, 0, 0)$	always	unstable
$P_3$	$\left(1 - \frac{2}{3(2-\gamma)}, 0, 0\right)$	$\begin{cases} 0 < \gamma < \frac{4}{3}, \text{ or} \\ \frac{4}{3} < \gamma < \frac{5}{3} \end{cases}$	saddle
$R_1$	$(0, 0, 1)$	always	saddle
$P_4$	$(0, 1, 0)$	always	stable

<sup>a</sup> The stability is analyzed for the flow restricted to the invariant set  $\varphi = 0$ .

Quadratic gravity,  $F(R) = R + \alpha R^2$ , is equivalent to a non-minimally coupled scalar

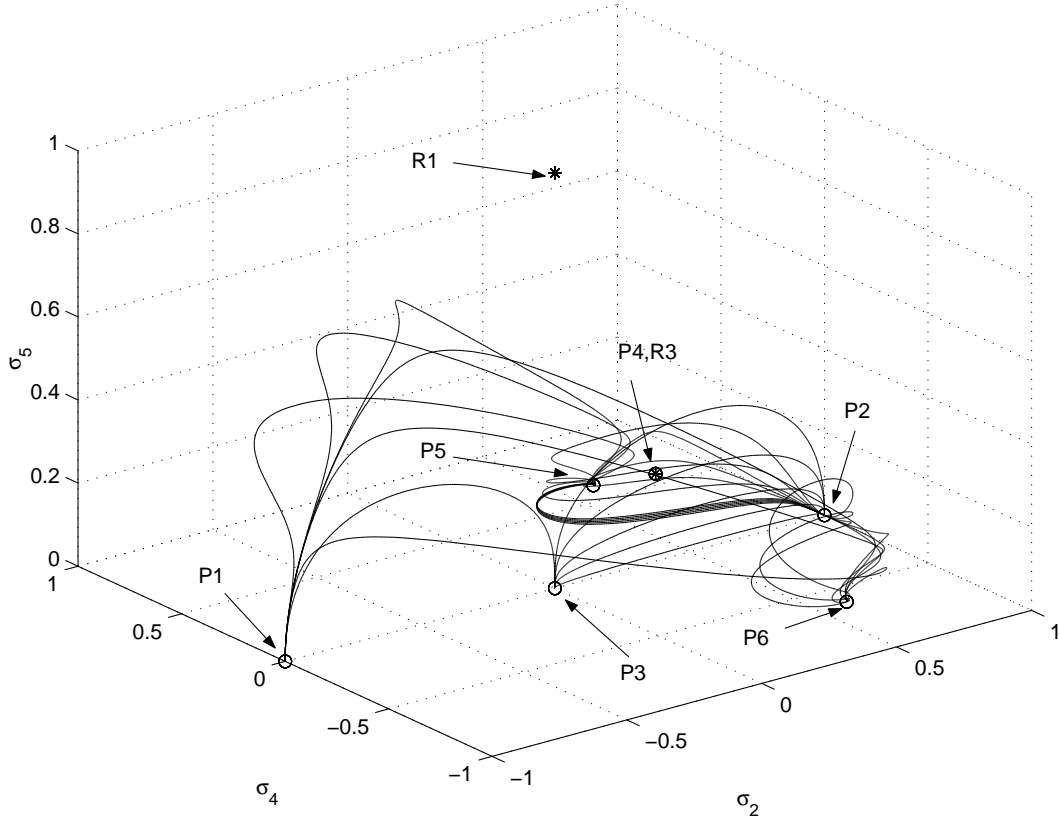


Figure 3.7: Some orbits in the invariant set  $\sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1$  for the choice of  $\varphi = 0$  for the model with coupling function (3.183) potential (3.66). We select the values of the parameters:  $\gamma = 1$ ,  $\epsilon = 1.00$ ,  $\mu = 2.00$ ,  $A = 0.50$ ,  $\alpha = 0.33$ ,  $B = 0.5$ , and  $\phi_0 = 0$ .

field with the potential

$$V(\phi) = \frac{1}{8\alpha} \left(1 - e^{-\sqrt{2/3}\phi}\right)^2 \quad (3.190)$$

and coupling function

$$\chi(\phi) = e^{\sqrt{\frac{2}{3}}\phi}. \quad (3.191)$$

Observe first that

$$W_\chi(\phi) = \chi'(\phi)/\chi(\phi) - \sqrt{2/3} = 0 \Rightarrow \lim_{\phi \rightarrow +\infty} W_\chi(\phi) = 0 \quad (3.192)$$

and

$$W_V(\phi) = V'(\phi)/V(\phi) = -\left(\sqrt{8/3}\right) / \left(1 - e^{\sqrt{\frac{2}{3}}\phi}\right) \Rightarrow \lim_{\phi \rightarrow +\infty} W_V(\phi) = 0. \quad (3.193)$$

In other words, the coupling function (3.191) and the potential (3.190) are WBI of exponential orders  $M = \sqrt{2/3}$  and  $N = 0$ , respectively.

It is easy to prove that the coupling function (3.191) and the potential (3.190) are at

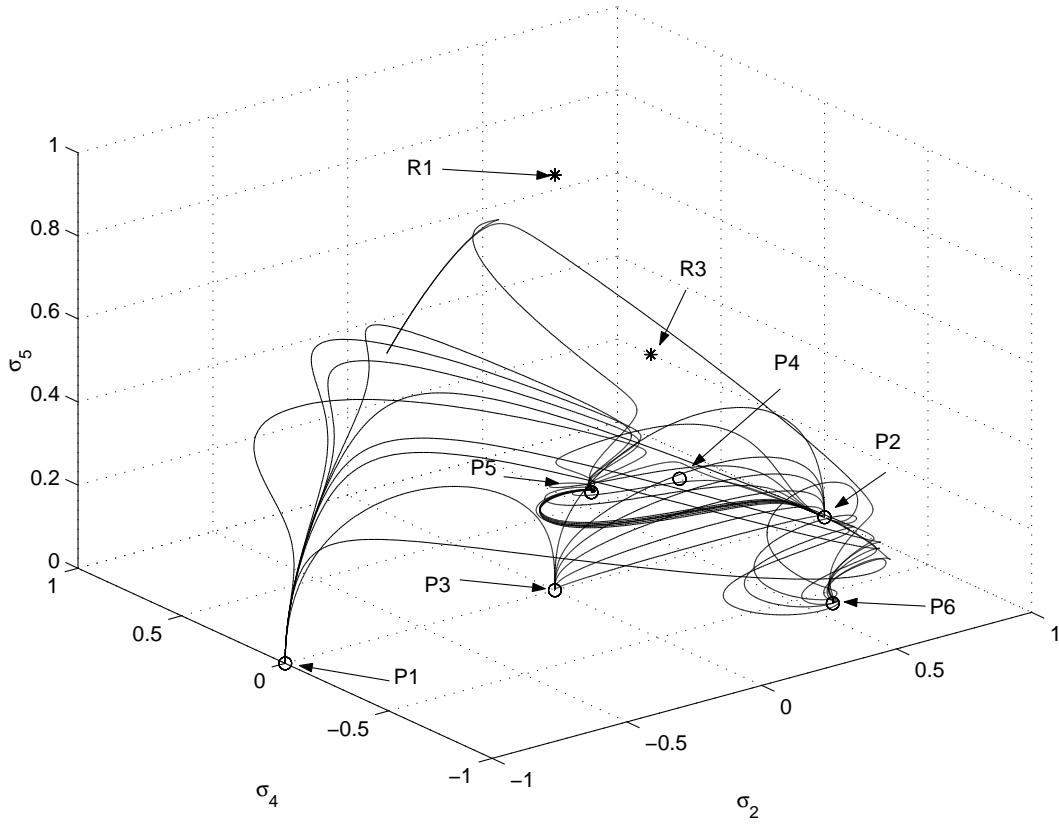


Figure 3.8: Some orbits in the invariant set  $\sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1$  for the choice of  $\varphi = 0$  for the model with coupling function (3.183) potential (3.66). We select the values of the parameters:  $\gamma = 1$ ,  $\epsilon = 1.00$ ,  $\mu = 2.10$ ,  $A = 0.50$ ,  $\alpha = 0.33$ ,  $B = 0.5$ , and  $\phi_0 = 0$ .

least  $\mathcal{E}_+^2$ , under the admissible coordinate transformation

$$\varphi = \phi^{-1} = f(\phi). \quad (3.194)$$

Using the coordinate transformation (3.194) we find

$$\overline{W}_\chi(\varphi) = 0. \quad (3.195)$$

$$\overline{W}_V(\varphi) = \begin{cases} -\frac{2\sqrt{\frac{2}{3}}}{1-e^{\sqrt{\frac{2}{3}}/\varphi}} & , \quad \varphi > 0 \\ 0 & , \quad \varphi = 0 \end{cases} \quad (3.196)$$

and

$$\overline{f}'(\varphi) = \begin{cases} -\varphi^2 & , \quad \varphi > 0 \\ 0 & , \quad \varphi = 0 \end{cases} \quad (3.197)$$

In this example, the evolution equations for  $\varphi$ ,  $\sigma_2$ ,  $\sigma_4$ , and  $\sigma_5$  are given by the equations (3.171)-(3.174) with  $M = \sqrt{2/3}$  and  $N = 0$ , and  $\overline{W}_\chi$ ,  $\overline{W}_V$ , and  $\overline{f}'$ , given respec-

tively by (3.195), (3.196) and (3.197). The state space is defined by

$$\Omega_\epsilon = \{(\varphi, \sigma_2, \sigma_4, \sigma_5) \in \mathbb{R}^4 : 0 \leq \varphi \leq \epsilon, \sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1, \sigma_4 \geq 0, \sigma_5 \geq 0\}.$$

Let us analyze the local stability of the singular points of the corresponding system. In the above analysis we are not taking into account perturbations in the  $\varphi$ -axis. It is obvious, from the previous analysis, that the center manifold of these singular points contains the  $\varphi$ -axis as a proper eigenvector. In table 3.7 are summarized the location, existence conditions and stability<sup>16</sup> of the singular points.

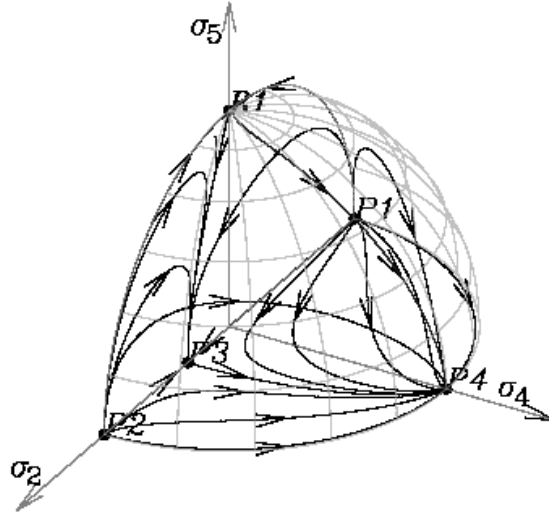


Figure 3.9: Projection of some orbits of (3.171)-(3.174) in the invariant set  $\varphi = 0$  for the coupling function (3.191) and the potential (3.190) for  $\gamma = 1$ . Observe that  $P_1$  and  $P_2$  are local sources,  $P_1$ , and  $P_3$  are saddles ( $P_3$  is the local attractor in the invariant set  $y = 0$ ) and  $P_4$  (the de Sitter solution) is the local sink in the invariant set  $\varphi = 0$ .

Let us discuss the stability properties of the singular points displayed in table 3.7.

The singular point  $P_1$  always exists. Its unstable manifold is 3D provided  $0 < \gamma < \frac{4}{3}$  or  $\frac{4}{3} < \gamma < \frac{5}{3}$ . Otherwise its unstable manifold is lower dimensional.

The singular point  $P_2$  always exists. It has a 3D unstable manifold. Although non-hyperbolic our numerical experiments suggest that it is a local source.

The singular point  $P_3$  exists for  $0 < \gamma < \frac{4}{3}$  or  $\frac{4}{3} < \gamma < \frac{5}{3}$  and it is neither a sink nor a local source.

<sup>16</sup>The stability is analyzed for the flow restricted to the invariant set  $\varphi = 0$ .

The singular point  $R_1$  always exists and it is neither a sink nor a local source.

The singular point  $P_4$  (corresponding to the *de Sitter* solution) always exists. Its stable manifold is 3D. Since  $P_4$  is nonhyperbolic the linear stability analysis is not conclusive. Thus we need to resort to numerical experimentation or alternatively we can use more sophisticated techniques such as normal forms expansion or center manifold theorem. Due its relevance, the full stability analysis of  $P_4$  is deserved to section 3.6.3.1

Let us discuss some physical properties of the cosmological solutions associated to the singular points displayed in table 3.7.

- $P_{1,2}$  represent kinetic-dominated cosmological solutions. They behave as stiff-like matter. The associated cosmological solution satisfies  $H = \frac{1}{3t-c_1}$ ,  $a = \sqrt[3]{3t-c_1}c_2$ ,  $\phi = c_3 \pm \sqrt{\frac{2}{3}} \ln(3t-c_1)$ , where  $c_j$ ,  $j = 1, 2, 3$  are integration constants. These solutions are associated with the local past attractors of the systems for an open set of values of the parameter  $\gamma$ .
- $P_3$  represents matter-kinetic scaling cosmological solutions such that  $H = \frac{3(\gamma-2)}{t(3\gamma-8)-3(\gamma-2)c_1}$ ,  $a = (t(3\gamma-8)-3(\gamma-2)c_1)^{1+\frac{2}{3\gamma-8}}c_2$ ,  $\rho = \frac{60-36\gamma}{(t(3\gamma-8)-3(\gamma-2)c_1)^2} + c_3$ , and  $\phi = c_4 + \frac{\sqrt{6}(3\gamma-4) \ln(t(3\gamma-8)-3(\gamma-2)c_1)}{3\gamma-8}$  where  $c_j$ ,  $j = 1, 2, 3, 4$  are integration constants.
- $R_1$  represents a radiation-dominated cosmological solutions satisfying  $H = \frac{1}{2t-c_1}$ ,  $a = \sqrt{2t-c_1}c_2$ ,  $\rho_r = \frac{3}{(2t-c_1)^2} + c_3$ .
- $P_4$  represents a de Sitter solution with  $H = \sqrt{\frac{V_0}{3}}$ ,  $a = c_1 \exp\left[\sqrt{\frac{V_0}{3}}t\right]$ ,  $V(\phi) = V_0$ .

In the figure 3.9 are displayed typical orbits of (3.171)-(3.174) in the invariant set  $\varphi = 0$ . The singular points  $P_1$  and  $P_2$  are local sources,  $R_1$ , and  $P_3$  are saddles ( $P_3$  is the local attractor in the invariant set  $y = 0$ ) and  $P_4$  (the de Sitter solution) is the local attractor in the invariant set  $\varphi = 0$ . However, concerning the full dynamics, it is locally unstable as we prove in next section by explicit calculation of the center manifold at  $P_4$ .

### 3.6.3.1 Stability Analysis of the *de Sitter* Solution in Quadratic Gravity

In order to analyze the stability of *de Sitter* solution we can use center manifold theorem. Let us proceed as follows. First, in order to remove the transcendental function in  $\overline{W}_V$ , let us introduce the new variable

$$u = \frac{1}{1 - \exp\left[\sqrt{\frac{2}{3}}/\varphi\right]},$$

taking values in the range

$$\frac{1}{1 - \exp\left[\sqrt{\frac{2}{3}}/\epsilon\right]} \leq u \leq 0.$$

In this way we obtain the new system of ordinary differential equations

$$\begin{aligned}
u' &= \frac{2u^2\sigma_2}{3} - \frac{2u\sigma_2}{3}, \\
\sigma_2' &= \frac{2u\sigma_4^2}{3} + \left(1 - \frac{\gamma}{2}\right)\sigma_2^3 + \frac{1}{6}(3\gamma - 4)\sigma_2^2 \\
&\quad + \sigma_2 \left(-\frac{\gamma\sigma_4^2}{2} + \frac{1}{6}(4 - 3\gamma)\sigma_5^2 + \frac{\gamma - 2}{2}\right) \\
&\quad + \frac{1}{6}(3\gamma - 4)\sigma_4^2 + \frac{1}{6}(3\gamma - 4)\sigma_5^2 + \frac{1}{6}(4 - 3\gamma), \\
\sigma_4' &= -\frac{2u\sigma_2\sigma_4}{3} + \left(1 - \frac{\gamma}{2}\right)\sigma_2^2\sigma_4 - \frac{\gamma\sigma_4^3}{2} + \sigma_4 \left(\frac{1}{6}(4 - 3\gamma)\sigma_5^2 + \frac{\gamma}{2}\right), \\
\sigma_5' &= \left(1 - \frac{\gamma}{2}\right)\sigma_2^2\sigma_5 - \frac{1}{2}\gamma\sigma_4^2\sigma_5 + \frac{1}{6}(4 - 3\gamma)\sigma_5^3 + \frac{1}{6}(3\gamma - 4)\sigma_5
\end{aligned} \tag{3.198}$$

describing the dynamics of quadratic gravity as  $\phi \rightarrow +\infty$ .

**Proposition 15** *The singular point  $q := (u, \sigma_2, \sigma_4, \sigma_5) = (0, 0, 1, 0)$  of the system (3.198) is locally unstable.*

In order to determine the local center manifold of (3.198) at  $q$  we have to transform the system into a form suitable for the application of the center manifold theorem (see section 2.2.5.3 for a summary of the techniques involved in the proof).

**Proof.**

**Case  $\gamma \neq 1$**

Let be  $\gamma \neq 1$ . The Jacobian of (3.198) at  $q = (0, 0, 1, 0)$  has eigenvalues  $0, -1, -\frac{2}{3}$ , and  $-\gamma$  with corresponding eigenvectors  $(0, 0, 0, 1)^T, (0, 0, \frac{3}{2}, 0)^T, \left(1, 0, 1, \frac{1}{3(\gamma-1)} - 1\right)^T$ , and  $(0, 1, 0, 0)^T$ . We shift the singular point to the origin by setting  $\hat{\sigma}_4 = \sigma_4 - 1$ . In order to transform the linear part of the vector field into Jordan canonical form, we define new variables  $(x, y_1, y_2, y_3) \equiv \mathbf{x}$ , by the equations

$$\begin{aligned}
u &= \frac{3x}{2}, \quad \sigma_2 = x + y_1 + y_3 \left(\frac{1}{3(\gamma-1)} - 1\right), \\
\hat{\sigma}_4 &= y_3, \quad \sigma_5 = y_2,
\end{aligned}$$

so that

$$\begin{pmatrix} x' \\ y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & -\gamma \end{pmatrix} \begin{pmatrix} x \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} f(x, y_1, y_2, y_3) \\ g_1(x, y_1, y_2, y_3) \\ g_2(x, y_1, y_2, y_3) \\ g_3(x, y_1, y_2, y_3) \end{pmatrix} \tag{3.199}$$



where

$$\begin{aligned}
f(x, y_1, y_2, y_3) &= x^3 + x^2 y_1 + \frac{x^2 y_3}{3(\gamma-1)} - x^2 y_3 - \frac{2x^2}{3} - \frac{2xy_1}{3} - \frac{2xy_3}{9(\gamma-1)} + \frac{2xy_3}{3}, \\
g_1(x, y_1, y_2, y_3) &= -\frac{x^3 \gamma}{2} - \frac{3}{2} x^2 y_1 \gamma + 2x^2 y_1 + x^2 y_3 \gamma + \frac{x^2 y_3}{3(\gamma-1)} - \frac{7x^2 y_3}{3} + \frac{x^2 \gamma}{6(\gamma-1)} - \frac{3}{2} x y_1^2 \gamma + \\
& 3x y_1^2 + 2x y_1 y_3 \gamma + \frac{x y_1 y_3}{\gamma-1} - \frac{17x y_1 y_3}{3} + \frac{2x y_1}{3} - \frac{1}{2} x y_2^2 \gamma + \frac{2x y_2^2}{3} - \frac{x y_3^2 \gamma^3}{(\gamma-1)^2} + \frac{16x y_3^2 \gamma^2}{3(\gamma-1)^2} - \frac{157x y_3^2 \gamma}{18(\gamma-1)^2} + \\
& \frac{41x y_3^2}{9(\gamma-1)^2} - x y_3 \gamma + \frac{2x y_3}{9(\gamma-1)} + \frac{4x y_3}{3} - \frac{y_1^3 \gamma}{2} + y_1^3 + y_1^2 y_3 \gamma + \frac{y_1^2 y_3}{3(\gamma-1)} - \frac{7y_1^2 y_3}{3} - \frac{3y_1^2 \gamma}{6-6\gamma} + \frac{4y_1^2}{6-6\gamma} - \\
& \frac{1}{2} y_1 y_2^2 \gamma + \frac{2y_1 y_2^2}{3} - \frac{y_1 y_3^2 \gamma^3}{(\gamma-1)^2} + \frac{10y_1 y_3^2 \gamma^2}{3(\gamma-1)^2} - \frac{73y_1 y_3^2 \gamma}{18(\gamma-1)^2} + \frac{16y_1 y_3^2}{9(\gamma-1)^2} - \frac{y_1 y_3 \gamma^3}{(\gamma-1)^2} + \frac{y_1 y_3 \gamma^2}{(\gamma-1)^2} + \frac{5y_1 y_3 \gamma}{3(\gamma-1)^2} - \\
& \frac{16y_1 y_3}{9(\gamma-1)^2} + \frac{y_2^2}{18-18\gamma} + \frac{y_2^2}{6} - \frac{y_3^2 \gamma^2}{3(\gamma-1)^3} + \frac{5y_3^2 \gamma}{6(\gamma-1)^3} - \frac{14y_3^2}{27(\gamma-1)^3}, \\
g_2(x, y_1, y_2, y_3) &= -\frac{1}{2} x^2 y_2 \gamma + x^2 y_2 - x y_1 y_2 \gamma + 2x y_1 y_2 + x y_2 y_3 \gamma - \frac{x y_2 y_3 \gamma}{3(\gamma-1)} + \frac{2x y_2 y_3}{3(\gamma-1)} - \\
& 2x y_2 y_3 - \frac{1}{2} y_1^2 y_2 \gamma + y_1^2 y_2 + y_1 y_2 y_3 \gamma - \frac{y_1 y_2 y_3 \gamma}{3(\gamma-1)} + \frac{2y_1 y_2 y_3}{3(\gamma-1)} - 2y_1 y_2 y_3 - \frac{y_2^3 \gamma}{2} + \frac{2y_2^3}{3} - y_2 y_3^2 \gamma + \\
& \frac{y_2 y_3^2 \gamma}{3(\gamma-1)} - \frac{y_2 y_3^2 \gamma}{18(\gamma-1)^2} - \frac{2y_2 y_3^2}{3(\gamma-1)} + \frac{y_2 y_3^2}{9(\gamma-1)^2} + y_2 y_3^2 - y_2 y_3 \gamma \text{ and} \\
g_3(x, y_1, y_2, y_3) &= -\frac{1}{2} x^2 y_3 \gamma - \frac{x^2 \gamma}{2} - x y_1 y_3 \gamma + x y_1 y_3 - x y_1 \gamma + x y_1 + x y_3^2 \gamma - \frac{x y_3^2 \gamma}{3(\gamma-1)} + \\
& \frac{x y_3^2}{3(\gamma-1)} - x y_3^2 + x y_3 \gamma - \frac{x y_3 \gamma}{3(\gamma-1)} + \frac{x y_3}{3(\gamma-1)} - x y_3 - \frac{1}{2} y_1^2 y_3 \gamma + y_1^2 y_3 - \frac{y_1^2 \gamma}{2} + y_1^2 + y_1 y_3^2 \gamma - \\
& \frac{y_1 y_3^2 \gamma}{3(\gamma-1)} + \frac{2y_1 y_3^2}{3(\gamma-1)} - 2y_1 y_3^2 + y_1 y_3 \gamma - \frac{y_1 y_3 \gamma}{3(\gamma-1)} + \frac{2y_1 y_3}{3(\gamma-1)} - 2y_1 y_3 - \frac{1}{2} y_2^2 y_3 \gamma + \frac{2y_2^2 y_3}{3} - \frac{y_2^2 \gamma}{2} + \frac{2y_2^2}{3} + \\
& \frac{y_3^3 \gamma}{3(\gamma-1)} - \frac{y_3^3 \gamma}{18(\gamma-1)^2} - y_3^3 \gamma - \frac{2y_3^3}{3(\gamma-1)} + \frac{y_3^3}{9(\gamma-1)^2} + y_3^3 + \frac{y_3^2 \gamma}{3(\gamma-1)} - \frac{y_3^2 \gamma}{18(\gamma-1)^2} - 2y_3^2 \gamma - \frac{2y_3^2}{3(\gamma-1)} + \frac{y_3^2}{9(\gamma-1)^2} + y_3^2.
\end{aligned}$$

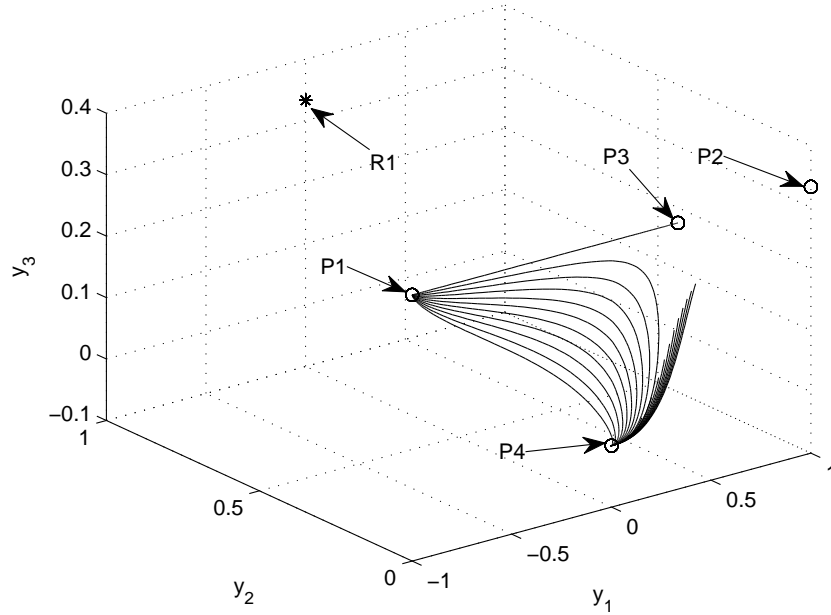


Figure 3.10: Projection of some orbits of (3.199) in the space  $y_1, y_2, y_3$  for the coupling function (3.191) and the potential (3.190) for  $\gamma = 1$ . The graphic shows the behavior in the stable manifold of  $P_4$ . The bulk of orbits in front of and at the right hand side of the figure represents a projection of the center(s) manifold(s).

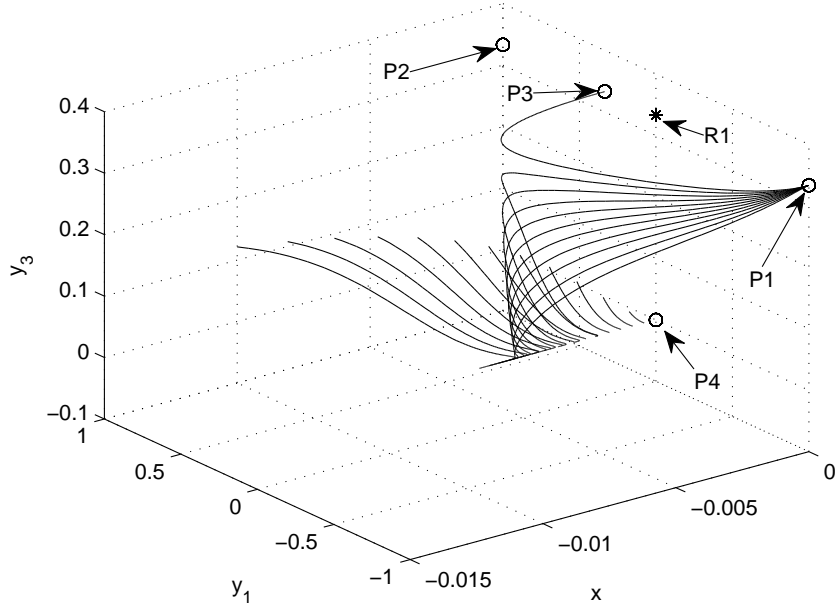


Figure 3.11: Projection of some orbits of (3.199) in the space  $x, y_1, y_3$  for the coupling function (3.191) and the potential (3.190) for  $\gamma = 1$ . The graphic shows the unstable character of  $P_4$  (trajectories starting at  $x < 0$  move away from the origin).

The system (3.199) is written in diagonal form

$$\begin{aligned} x' &= Cx + f(x, \mathbf{y}) \\ \mathbf{y}' &= P\mathbf{y} + \mathbf{g}(x, \mathbf{y}), \end{aligned} \quad (3.200)$$

where  $(x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^3$ ,  $C$  is the zero  $1 \times 1$  matrix,  $P$  is a  $3 \times 3$  matrix with negative eigenvalues and  $f, \mathbf{g}$  vanish at  $\mathbf{0}$  and have vanishing derivatives at  $\mathbf{0}$ . The center manifold theorem 13 asserts that there exists a 1-dimensional invariant local center manifold  $W^c(\mathbf{0})$  of (3.200) tangent to the center subspace (the  $\mathbf{y} = \mathbf{0}$  space) at  $\mathbf{0}$ . Moreover,  $W^c(\mathbf{0})$  can be represented as

$$W^c(\mathbf{0}) = \{(x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^3 : \mathbf{y} = \mathbf{h}(x), |x| < \delta\}; \mathbf{h}(0) = \mathbf{0}, D\mathbf{h}(0) = \mathbf{0},$$

for  $\delta$  sufficiently small (see definition 13). The restriction of (3.200) to the center manifold is (see definition 2.36)

$$x' = f(x, \mathbf{h}(x)). \quad (3.201)$$

According to Theorem 14, if the origin  $x = 0$  of (3.201) is stable (asymptotically stable) (unstable) then the origin of (3.200) is also stable (asymptotically stable) (unstable). Therefore, we have to find the local center manifold, i.e., the problem reduces to the computation of  $\mathbf{h}(x)$ .

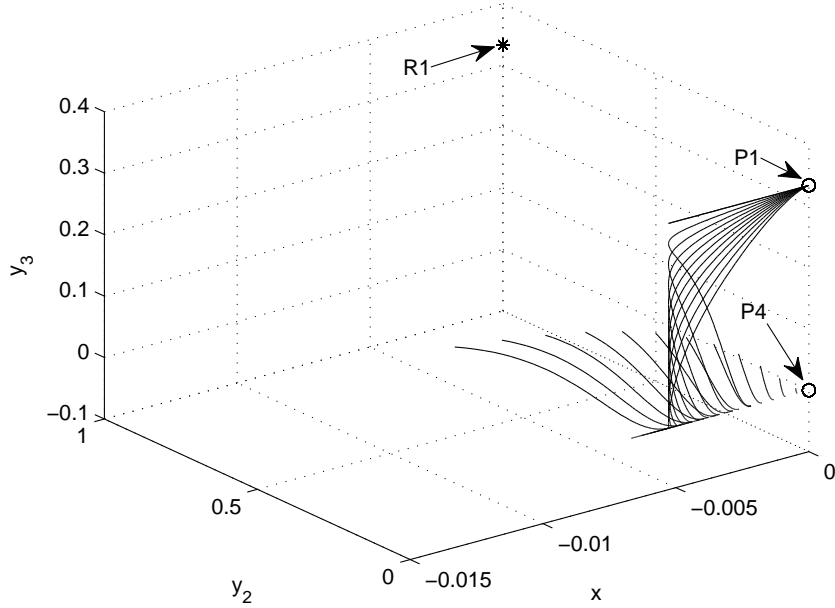


Figure 3.12: Projection of some orbits of (3.199) in the space  $x, y_2, y_3$  for the coupling function (3.191) and the potential (3.190) for  $\gamma = 1$ . The graphic shows the unstable character of  $P_4$  (the orbits depart from the origin for  $x < 0$ ).

Substituting  $\mathbf{y} = \mathbf{h}(x)$  in the second component of (3.200) and using the chain rule,  $\mathbf{y}' = D\mathbf{h}(x) x'$ , one can show that the function  $\mathbf{h}(x)$  that defines the local center manifold satisfies

$$D\mathbf{h}(x) [f(x, \mathbf{h}(x))] - P\mathbf{h}(x) - \mathbf{g}(x, \mathbf{h}(x)) = 0. \quad (3.202)$$

According to Theorem 15, equation (3.202) can be solved approximately by using an approximation of  $\mathbf{h}(x)$  by a Taylor series at  $x = 0$ . Since  $\mathbf{h}(0) = \mathbf{0}$  and  $D\mathbf{h}(0) = \mathbf{0}$ , it is obvious that  $\mathbf{h}(x)$  commences with quadratic terms. We substitute

$$\mathbf{h}(x) =: \begin{bmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \end{bmatrix} = \begin{bmatrix} a_1x^2 + a_2x^3 + O(x^4) \\ b_1x^2 + b_2x^3 + O(x^4) \\ c_1x^2 + c_2x^3 + O(x^4) \end{bmatrix}$$

into (3.202) and set the coefficients of like powers of  $x$  equal to zero to find the unknowns  $a_1, b_1, c_1, \dots$

Since  $y_2$  is absent from the first of (3.200), we give only the result for  $h_1(x)$  and  $h_3(x)$ . We find  $a_1 = \frac{\gamma}{6(\gamma-1)}$ ,  $a_2 = -\frac{3\gamma-5}{9(\gamma-1)}$ ,  $c_1 = -\frac{1}{2}$ ,  $c_2 = -\frac{2}{3}$ .<sup>17</sup> Therefore, (3.201) yields

$$x' = -\frac{2x^2}{3} + \frac{5x^3}{9} + \frac{4x^4}{9} + O(x^5). \quad (3.203)$$

<sup>17</sup>We find  $b_1 = b_2 = 0$ .

It is obvious that the origin  $x = 0$  of (3.203) is locally unstable (saddle point). Hence, the origin  $\mathbf{x} = \mathbf{0}$  of the full four-dimensional system is unstable.

**Case  $\gamma = 1$**

Let be  $\gamma = 1$ . The Jacobian of (3.198) at  $q = (0, 0, 1, 0)$  has eigenvalues  $-1, -1, -\frac{2}{3}$ , and  $0$  with corresponding eigenvectors  $(0, 0, 0, \frac{3}{2})^T, (1, 0, 0, 1)^T, (0, -3, 0, 0)^T$  and  $(0, 0, 1, 0)^T$ . As before we shift the singular point to the origin by setting  $\hat{\sigma}_4 = \sigma_4 - 1$  and define new variables  $(x, y_1, y_2, y_3) \equiv \mathbf{x}$ , by the equations

$$\begin{aligned} u &= \frac{3x}{2}, \\ \sigma_2 &= x + y_1, \\ \hat{\sigma}_4 &= -3y_3, \\ \sigma_5 &= y_2 \end{aligned}$$

so that

$$\begin{pmatrix} x' \\ y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} f(x, y_1, y_2, y_3) \\ g_1(x, y_1, y_2, y_3) \\ g_2(x, y_1, y_2, y_3) \\ g_3(x, y_1, y_2, y_3) \end{pmatrix}, \quad (3.204)$$

where  $f(x, y_1, y_2, y_3) = x^3 + x^2 y_1 - \frac{2x^2}{3} - \frac{2xy_1}{3}$ ,  
 $g_1(x, y_1, y_2, y_3) = -\frac{x^3}{2} + \frac{x^2 y_1}{2} + \frac{x^2}{2} + \frac{3xy_1^2}{2} + \frac{xy_1}{3} + \frac{xy_2^2}{6} + \frac{9xy_3^2}{2} - 3xy_3 + \frac{y_1^3}{2} - \frac{y_1^2}{6} + \frac{y_1 y_2^2}{6} - \frac{9y_1 y_3^2}{2} + 3y_1 y_3 - \frac{y_2^2}{6} - \frac{3y_3^2}{2}$ ,  
 $g_2(x, y_1, y_2, y_3) = \frac{x^2 y_2}{2} + xy_1 y_2 + \frac{y_1^2 y_2}{2} + \frac{y_2^3}{6} - \frac{9y_2 y_3^2}{2} + 3y_2 y_3$  and  
 $g_3(x, y_1, y_2, y_3) = -\frac{x^2 y_3}{2} + \frac{x^2}{6} + \frac{y_1^2 y_3}{2} - \frac{y_1^2}{6} + \frac{y_2^2 y_3}{6} - \frac{y_2^2}{18} - \frac{9y_3^3}{2} + \frac{9y_3^2}{2}$ .

Observe that the system (3.204) is now in the canonical form (3.200). Then, we proceed to the calculation of the center manifold. The procedure is fairly systematic and since we present it completely in the previous analysis we consider do not repeat it here. Instead, we present the relevant calculations. We obtain  $a_1 = \frac{2}{3}, a_2 = \frac{1}{3}, b_1 = 0, b_2 = 0, c_1 = \frac{1}{6}, c_2 = \frac{2}{9}$  for the Taylor expansion coefficients of

$$\mathbf{h}(x) =: \begin{bmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \end{bmatrix} = \begin{bmatrix} a_1 x^2 + a_2 x^3 + O(x^4) \\ b_1 x^2 + b_2 x^3 + O(x^4) \\ c_1 x^2 + c_2 x^3 + O(x^4) \end{bmatrix}.$$

By substituting this values of the unknowns  $a_1, b_1, c_1, \dots$  we obtain that, for  $\gamma = 1$ , the dynamics of the center manifold in given also by equation (3.203). The conclusion is

straightforward: the origin  $x = 0$  of (3.203) is locally unstable (saddle point). Hence, the origin  $\mathbf{x} = \mathbf{0}$  of the full four-dimensional system is unstable.

This completes the Proof. ■

The result of proposition 15 complements the result of the proposition discussed in [405] p. 5, where it was proved the local asymptotic instability of the de Sitter universe for positively curved FRW models with a perfect fluid matter source and a scalar field which arises in the conformal frame of the  $R + \alpha R^2$  theory.

### 3.6.4 $R^n$ -Gravity

Let us consider the model with  $F(R) = R^n$  where we have re-scaled the usual multiplicative constant. It can be proved that, for  $n > 1$ ,  $R^n$ -gravity is conformally equivalent to a non-minimally coupled scalar field with a positive potential

$$V(\phi) = r(n)e^{\lambda(n)\phi} \quad (3.205)$$

where  $r(n) = \frac{1}{2}(n-1)n^{-\frac{n}{n-1}}$  and  $\lambda(n) = -\frac{\sqrt{\frac{2}{3}}(n-2)}{n-1}$ , with coupling function given by (3.191).

In this example, the evolution equations for  $\varphi$ ,  $\sigma_2$ ,  $\sigma_4$ , and  $\sigma_5$  are given by the equations (3.171)-(3.174) with  $M = \sqrt{2/3}$ ,  $N = -\frac{\sqrt{\frac{2}{3}}(n-2)}{n-1}$ ,  $\overline{W}_\chi(\varphi) = \overline{W}_V(\varphi) = 0$ , and  $\overline{f}'$ , given by (3.197). The state space is defined by

$$\Omega_\epsilon = \{(\varphi, \sigma_2, \sigma_4, \sigma_5) \in \mathbb{R}^4 : 0 \leq \varphi \leq \epsilon, \sigma_2^2 + \sigma_4^2 + \sigma_5^2 \leq 1, \sigma_4 \geq 0, \sigma_5 \geq 0\}.$$

In table 3.8 and 3.9 are summarized the location, existence conditions and stability of the singular points.

Let us discuss the stability properties of the singular points displayed in table 3.9.

The singular point  $P_1$  always exists. Its unstable manifold is 3D provided  $0 < \gamma < \frac{4}{3}$ ,  $n > \frac{5}{4}$  or  $\frac{4}{3} < \gamma < 2$ ,  $n > \frac{5}{4}$ . Otherwise its unstable manifold is lower-dimensional.

The singular point  $P_3$  always exists and it is a local source.

The singular point  $P_3$  exists for  $0 < \gamma < \frac{4}{3}$  or  $\frac{4}{3} < \gamma < 2$ . It is neither a source nor a sink.

The singular point  $R_1$  always exists and it is neither a source nor a sink.

The singular point  $P_4$  exists for  $n > \frac{5}{4}$ . Its stable manifold is 3D if  $\gamma \neq \frac{4}{3}$ ,  $n > 2$  or  $n_+ < n \leq 2$ ,  $\Gamma(n) < \gamma < \frac{4}{3}$  or  $n_+ < n \leq 2$ ,  $\frac{4}{3} < \gamma < 2$ . Where we have defined the grouping constants  $n_+ = \frac{1}{5}(4 + \sqrt{6})$ ,  $\Gamma(n) = -\frac{2n(n-2)}{3-9n+6n^2}$ . Otherwise its stable manifold is lower-dimensional.

The singular point  $R_3$  exists for  $1 < n < n_+$ . Thus  $P_4$  and  $R_3$  are in the same phase portrait for  $\frac{5}{4} < n < n_+$ .  $R_3$  admits a 3D stable manifold for  $\frac{4}{3} < \gamma < 2$ ,  $N_+ \leq n < n_+$

Table 3.8: Location of the singular points of the flow of (3.171)-(3.174) defined in the invariant set  $\{p \in \Omega_\epsilon : \varphi = 0\}$  for  $M = \sqrt{2/3}$  and  $N = -\frac{\sqrt{\frac{2}{3}}(n-2)}{n-1}$ . We use the notations  $n_+ = \frac{1}{5}(4 + \sqrt{6})$ ,  $N_+ = \frac{2}{27}(11 + 2\sqrt{10})$ ,  $\Gamma(n) = -\frac{2n(n-2)}{3-9n+6n^2}$ , and  $\Gamma_+(\gamma) = \frac{9\gamma + \sqrt{9\gamma^2 + 48\gamma + 16} + 4}{12\gamma + 4}$ .

Label	$(\sigma_2, \sigma_4, \sigma_5)$	Existence
$P_1$	$(-1, 0, 0)$	always
$P_2$	$(1, 0, 0)$	always
$P_3$	$\left(\frac{4-3\gamma}{3(2-\gamma)}, 0, 0\right)$	$\begin{cases} 0 < \gamma < \frac{4}{3}, \text{ or} \\ \frac{4}{3} < \gamma < \frac{5}{3} \end{cases}$
$R_1$	$(0, 0, 1)$	always
$P_4$	$\left(\frac{n-2}{3(n-1)}, \sqrt{1 - \frac{(n-2)^2}{9(n-1)^2}}, 0\right)$	$n > \frac{5}{4}$
$R_3$	$\left(\frac{2(n-1)}{n-2}, -\frac{\sqrt{2}(n-1)}{n-2}, -\frac{\sqrt{(8-5n)n-2}}{n-2}\right)$	$1 < n < n_+$
$P_5$	$\left(\frac{3(n-1)\gamma}{3\gamma n - 2n - 3\gamma}, -\frac{\sqrt{2}\sqrt{n-1}\sqrt{4n-3\gamma}}{3\gamma n - 2n - 3\gamma}, 0\right)$	$\begin{cases} n = \frac{5}{4}, \gamma = \frac{5}{3} \text{ or} \\ 0 < \gamma < \frac{4}{3}, 1 < n \leq \Gamma_+(\gamma), \text{ or} \\ \frac{4}{3} < \gamma < \frac{5}{3}, \frac{3\gamma}{4} \leq n \leq \Gamma_+(\gamma). \end{cases}$
$P_6$	$\left(\frac{3(n-1)\gamma}{3\gamma n - 2n - 3\gamma}, \frac{\sqrt{2}\sqrt{n-1}\sqrt{4n-3\gamma}}{3\gamma n - 2n - 3\gamma}, 0\right)$	$\frac{4}{3} < \gamma \leq \frac{5}{3}, n = \frac{3\gamma}{4}$ <sup>a</sup>

<sup>a</sup> In this case  $P_6$  and  $P_3$  coincides. Thus, the singular points are nonhyperbolic. There exists a 1-dimensional stable manifold and a 1-dimensional unstable manifold provided  $\frac{4}{3} < \gamma < \frac{5}{3}$ .

or  $\frac{4}{3} < \gamma < 2, 1 < n < N_+$ , where we have defined  $N_+ = \frac{2}{27}(11 + 2\sqrt{10})$ . Otherwise its stable manifold is lower-dimensional.

The singular point  $P_5$  exists for  $n = \frac{5}{4}, \gamma = \frac{5}{3}$  or  $0 < \gamma < \frac{4}{3}, 1 < n \leq \Gamma_+(\gamma)$ , or  $\frac{4}{3} < \gamma < \frac{5}{3}, \frac{3\gamma}{4} \leq n \leq \Gamma_+(\gamma)$ , where we have defined  $\Gamma_+(\gamma) = \frac{9\gamma + \sqrt{9\gamma^2 + 48\gamma + 16} + 4}{12\gamma + 4}$ .  $P_5$  admits a 3D stable manifold provided  $\frac{41}{25} \leq n < 2, 0 < \gamma < \Gamma(n)$ , or  $1 < n \leq N_+, 0 < \gamma < \frac{4}{3}$ , or  $N_+ < n < \frac{41}{25}, 0 < \gamma < \frac{4\sqrt{96n^5 - 272n^4 + 230n^3 - 50n^2}}{3(4n^3 - 24n^2 + 29n - 9)} - \frac{2(10n^3 - 19n^2 + 13n)}{3(4n^3 - 24n^2 + 29n - 9)}$ . Otherwise its stable manifold is lower-dimensional.

The singular point  $P_6$  exists for  $\frac{4}{3} < \gamma \leq \frac{5}{3}, n = \frac{3\gamma}{4}$ . In this case  $P_6$  and  $P_3$  coincides; thus, the singular points are non-hyperbolic with a 2D center manifold, a 1D unstable manifold and a 1D stable manifold.

Let us discuss some physical properties of the cosmological solutions associated to the singular points displayed in table 3.8.

- $P_{1,2}$  represent kinetic-dominated cosmological solutions. They behave as stiff-

Table 3.9: Stability of the singular points of the flow of (3.171)-(3.174) defined in the invariant set  $\{p \in \Omega_\epsilon : \varphi = 0\}$  for  $M = \sqrt{2/3}$  and  $N = -\frac{\sqrt{2/3}(n-2)}{n-1}$ . We use the notations  $n_+ = \frac{1}{5}(4 + \sqrt{6})$ ,  $N_+ = \frac{2}{27}(11 + 2\sqrt{10})$ ,  $\Gamma(n) = -\frac{2n(n-2)}{3-9n+6n^2}$ , and  $\Gamma_+(\gamma) = \frac{9\gamma + \sqrt{9\gamma^2 + 48\gamma + 16} + 4}{12\gamma + 4}$ .

Label	Stability <sup>a</sup>
$P_1$	unstable for $0 < \gamma < \frac{4}{3}, n > \frac{5}{4}$ ; or $\frac{4}{3} < \gamma < \frac{5}{3}, n > \frac{5}{4}$ ; saddle, otherwise
$P_2$	unstable for $\gamma \neq \frac{4}{3}, n > 1$ saddle, otherwise
$P_3$	saddle
$R_1$	saddle
$P_4$	stable for $\left\{ \begin{array}{l} \gamma \neq \frac{4}{3}, n > 2 \text{ or} \\ n_+ < n \leq 2, \Gamma(n) < \gamma < \frac{4}{3} \text{ or} \\ n_+ < n \leq 2, \frac{4}{3} < \gamma < 2. \end{array} \right.$ saddle, otherwise
$R_3$	stable for $\left\{ \begin{array}{l} \frac{4}{3} < \gamma < 2, N_+ \leq n < n_+ \text{ or} \\ \frac{4}{3} < \gamma < 2, 1 < n < N_+. \end{array} \right.$ saddle, otherwise
$P_5$	stable for $\left\{ \begin{array}{l} \frac{41}{25} \leq n < 2, 0 < \gamma < \Gamma(n) \text{ or} \\ 1 < n \leq N_+, 0 < \gamma < \frac{4}{3} \text{ or} \\ N_+ < n < \frac{41}{25}, 0 < \gamma < \frac{4\sqrt{96n^5-272n^4+230n^3-50n^2}}{3(4n^3-24n^2+29n-9)} - \frac{2(10n^3-19n^2+13n)}{3(4n^3-24n^2+29n-9)}. \end{array} \right.$ saddle, otherwise

<sup>a</sup> The stability is analyzed for the flow restricted to the invariant set  $\varphi = 0$ .

like matter. The associated cosmological solution satisfies  $H = \frac{1}{3t-c_1}$ ,  $a = \sqrt[3]{3t-c_1}c_2$ ,  $\phi = c_3 \pm \sqrt{\frac{2}{3}} \ln(3t-c_1)$ , where  $c_j$ ,  $j = 1, 2, 3$  are integration constants. These solutions are associated with the local past attractors of the systems for an open set of values of the parameter  $\mu$ .

- $P_3$  represents matter-dominated cosmological solutions that satisfy  $H = \frac{2}{3t\gamma-2c_1}$ ,  $a = (3t\gamma-2c_1)^{\frac{2}{3\gamma}}c_2$ ,  $\rho = \frac{12}{(3t\gamma-2c_1)^2} + c_3$ .

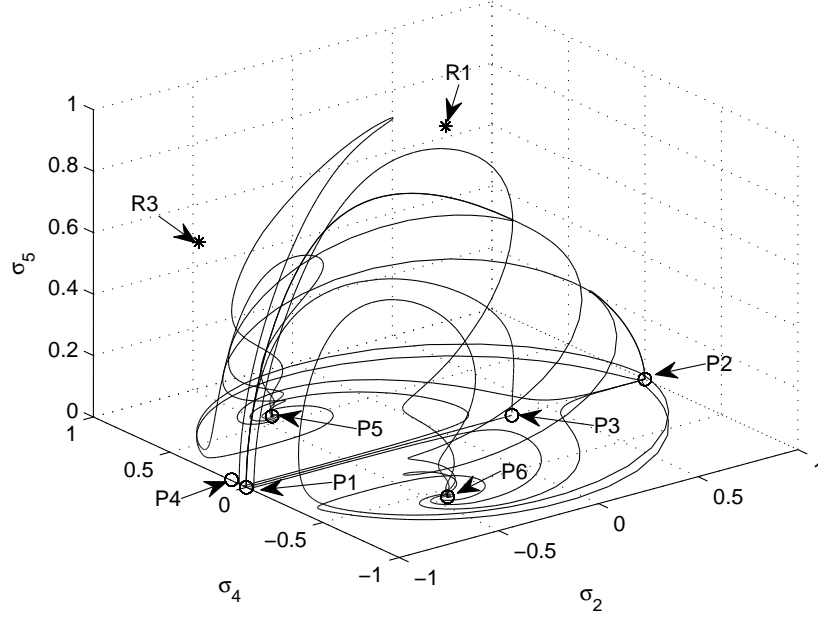


Figure 3.13: Projection in  $\varphi = 0$  of some orbits of the flow of (3.171)-(3.174) for  $M = \sqrt{2/3}$ ,  $N = -\frac{\sqrt{2/3}(n-2)}{n-1}$ . We set  $n = 1.251$ . Observe that  $R_1$ ,  $R_3$  are in the region of physical interest. These are saddle points. The singular points  $P_4$  and  $P_3$  exist and are saddle points.  $P_1$  and  $P_2$  are local sources and  $P_5$  is a local sink. We display some orbits in the halfspace  $\sigma_4 < 0$  (corresponding to contracting universes) for aesthetical purposes. Observe that  $P_6$  mirrors the behavior of  $P_5$ .

- $R_1$  represents a radiation-dominated cosmological solutions satisfying  $H = \frac{1}{2t-c_1}$ ,  $a = \sqrt{2t-c_1}c_2$ ,  $\rho_r = \frac{3}{(2t-c_1)^2} + c_3$ .
- $P_4$  represents power-law scalar-field dominated inflationary cosmological solutions. It is easy to obtain the asymptotic exact solution:  $H = \frac{3(n-1)^2}{(n-2)^2t-3(n-1)^2c_1}$ ,  $a = ((n-2)^2t-3(n-1)^2c_1)^{\frac{3(n-1)^2}{(n-2)^2}}c_3$ ,  $\phi = c_2 + \frac{\sqrt{6}(n-1)\ln((n-2)^2t-3(n-1)^2c_1)}{n-2}$ .
- $P_5$  represent matter-kinetic-potential scaling solutions satisfying  $H = \frac{3(n-1)\gamma-2n}{3(n-2)t\gamma+(n(2-3\gamma)+3\gamma)c_1}$ ,  $a = (3(n-2)t\gamma+(n(2-3\gamma)+3\gamma)c_1)^{\frac{-3\gamma n+2n+3\gamma}{6\gamma-3n\gamma}}c_2$ , and  $\rho = c_3 - \frac{6(3\gamma+n(-9\gamma+n(6\gamma+2)-4))}{(3(n-2)t\gamma+(n(2-3\gamma)+3\gamma)c_1)^2}$ ,  $\phi = c_4 + \frac{\sqrt{6}(n-1)\log(3(n-2)t\gamma+(n(2-3\gamma)+3\gamma)c_1)}{n-2}$ .
- $R_3$  represent radiation-kinetic-potential scaling solutions satisfying  $H = \frac{1}{2t-c_1}$ ,  $a = \sqrt{2t-c_1}c_2$ ,  $\rho_r = c_3 - \frac{3(n(5n-8)+2)}{(n-2)^2(c_1-2t)^2}$ ,  $\phi = c_4 + \frac{\sqrt{6}(n-1)\log(2t-c_1)}{n-2}$ .

To complete the section we present in figures 3.13, and 3.14 a numerical elaboration of the model under consideration.



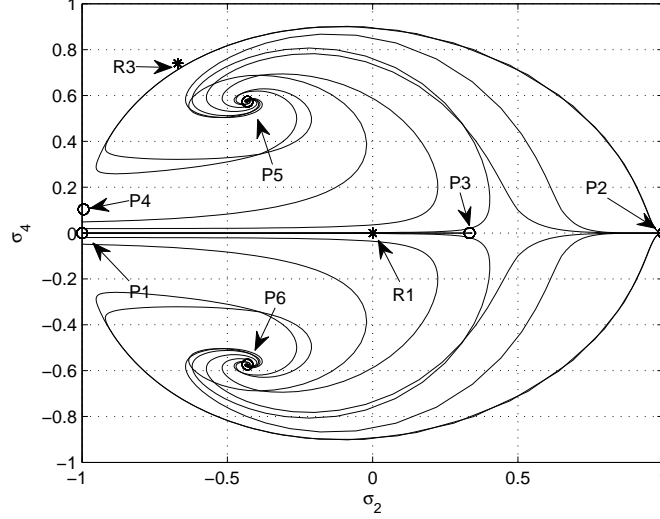


Figure 3.14: Projection of the orbits displayed in figure 3.13 to  $\sigma_5 = 0$ .

### 3.7 Conclusion

In this chapter we have extended several results about flat FRW models in the conformal (Einstein) frame in scalar-tensor gravity theories including  $f(R)$  theories through conformal transformation. Particularly we have considered a cosmological model based on the action

$$S_{EF} = \int_{M_4} d^4x \sqrt{|g|} \left\{ \frac{1}{2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \chi(\phi)^{-2} \mathcal{L}_m(\mu, \nabla\mu, \chi(\phi)^{-1}g_{\alpha\beta}) \right\},$$

where  $R$  is the curvature scalar,  $\phi$  is the a scalar field, and  $V$  is the potential of the scalar field and  $\chi$  is the coupling function. We have consider both ordinary matter described by a perfect fluid with equation of state  $p = (\gamma - 1)\rho$  (coupled to the scalar field) and radiation  $\rho_r$  in order to describe the dynamics in a cosmological epoch where matter and radiation coexisted.

We have considered scalar fields with arbitrary (positive) potentials and arbitrary coupling functions from the beginning. Then, we have straightforwardly introduced mild assumptions under such functions (differentiable class, number of singular points, asymptotes, etc.) in order to clarify the structure of the phase space of the dynamical system. We have obtained several analytical results. Also, we have presented several numerical evidences that confirm some of these results.

Our main results are the following.

1. We have proved the Proposition 4. This proposition states that if the potential is

nonnegative and has a local zero minimum at  $\phi = 0$ ; its derivative is bounded in the same set where the potential is; and provided the derivative of the logarithm of the coupling function has an upper bound, then the energy density of the background as well as the kinetic term tend to zero when the time goes forward. Thus, the total energy density of the universe will be dominated into the future by the potential energy of the scalar field and the universe would expand forever in a de Sitter phase. This result is an extension of the Proposition 2 in [404] to the non-minimal coupling context.

2. With the same hypotheses as in 4 and with the additional hypothesis of  $V(\phi)$  being strictly decreasing (increasing) for negative (positive) values of the scalar field, we have proved in 5 that the scalar field can be either  $-\infty$  or zero or  $+\infty$ . This means that under the above hypothesis the scalar field diverges into the future or it equals to zero (the last case holds only if the Hubble scalar vanish towards the future). This proposition is an extension of proposition 3 in [404].
3. Assuming that the potential is non-negative (with not necessarily a local minimum at  $(0, 0)$ ) having a continuous derivative (bounded on a set  $A$  provided the potential is bounded on it). Assuming also that the potential is strictly decreasing such that  $V(\phi(t))$ , viewed as a function of  $t$ , diverges when  $t$  approaches infinity. Then the cosmological model enters a de Sitter phase into the future characterized by the divergence of the scalar field. If additional the potential vanish asymptotically to the future as a function of  $\phi$ , the Hubble scalar vanishes too. This fact is true for the exponential potential. (This result is presented in proposition 6).
4. For the model including radiation we have formulated and proved proposition 7 generalizing analogous result in [407]. It states that if the potential  $V(\phi)$  is such that the (possibly empty set) where it is negative is bounded and the (possibly empty) set of singular points of  $V(\phi)$  is finite, then, the singular point

$$\mathbf{p}_* := \left( \phi_*, y_* = 0, \rho_* = 0, \rho_r = 0, H = \sqrt{\frac{V(\phi_*)}{3}} \right),$$

where  $\phi_*$  is a strict local minimum for  $V(\phi)$ , is an asymptotically stable singular point for the flow. From the physical viewpoint this result is relevant since it provides conditions for the asymptotic stability of the *de Sitter* solution.

5. After the introduction of modified normalized variables, we have proved that the phase space of the model has the structure of a manifold with boundaries (see propositions 9 and 10 for the invariant set  $\rho_r = 0$  and propositions 11, 12 for the model including radiation). We have devised a monotonic function for the flow of the dy-

namical system which allow for the identification of some invariant sets (the more relevant invariant sets are discussed in proposition 8).

6. We have provided approximated center manifolds for the vector field around the inflection points and the strict degenerate local minimum of the potential. For inflection and degenerate local maximum points of order  $n = 2$  for the potential the center manifold of  $Q_2$  is locally unstable whereas for degenerate local minimum points of order  $n = 2$  for the potential it is locally asymptotically stable (see propositions 13 and 14). The results obtained are in agreement with the result in proposition 7.
7. In lemma 1 it is proved that the orbit passing through an arbitrary point  $p \in \Sigma$  (representing cosmological solutions with non-vanishing dimensionless background energy density and positive finite Hubble parameter) is past asymptotic to a regime where the Hubble parameter diverges containing an initial singularity into the past, and is future asymptotic to a regime where the background density is negligible into the future. This result is obtained by constructing a monotonic function defined on an invariant set (subset of the invariant set  $\rho_r = 0$ ) and by applying the LaSalle monotonicity principle (theorem 4.12, [418]).
8. When the scalar field is incorporated as a dynamical variable it typically diverges into the past. This fact (proved in [403]) is valid also to our general context. The theorem 22 is a generalization to the theorem 1 in the reference [403]. Theorem 22 states that if the potential and the coupling function are sufficiently smooth functions, then for almost all the points lying in the Hubble-normalized state space, the scalar field diverges when the orbit through  $p$  is followed backward in time. The demonstration of this theorem relies on the topological properties of the state space and the existence of monotonic functions.
9. We have proved theorem 26 which is a generalization of the related result in [136, 403, 404] and an extension of theorem 22. This result states that if  $\chi(\phi)$  and  $V(\phi)$  are positive functions of class  $C^3$ , such that  $\chi$  has at most a finite number of stationary points and does not tend to zero in any compact set of  $\mathbb{R}$ , then, given  $p$ , an interior point of the phase space manifold, the scalar field,  $\phi$ , is unbounded through the past orbit  $O^-(p)$ . The relevance from the physical viewpoint of this result is twofold. First, the inclusion of radiation in the cosmic budget does not influence radically the early-time behavior of the scalar field. This result is somewhat expected since for small scale factor  $a$ , the energy densities of radiation and the scalar field goes respectively as  $\rho_r \sim a^{-4}$ , and  $\rho_\phi \approx \frac{\dot{\phi}^2}{2} \sim a^{-6}$  (the last approximation is supported by theorems 4.1 and 4.2 in [136]). Second, the result of theorem 26 makes clear

that in order to investigate the generic past asymptotic dynamics of the flow we must scan the region of the phase space where  $|\phi| \rightarrow \infty$ .

10. For the analysis of the system as  $\phi \rightarrow \infty$  we have defined a suitable change of variables to bring a neighborhood of  $\phi = \infty$  in a bounded set. This method was first introduced in [403] (see also [414]). By assuming some regularity conditions on the potential and on the coupling function in that regime we have constructed a dynamical system (well suited to investigate the dynamics where the scalar field diverges, i.e. near the initial singularity) defined in the invariant set  $\rho_r = 0$ . The singular points therein are investigated and the cosmological solutions associated to them are characterized. We find the existence of three singular points  $P_3$ ,  $P_5$  and  $P_6$ . They are in the boundary of the phase space  $\Sigma_\epsilon$ . They represent cosmological scaling solutions (where the contribution of the dimensionless potential energy is negligible). By tuning the free parameters they can be accelerating. In contrast in the reference [403] there exists only one (in our notation,  $P_4$ ) representing an accelerating cosmology. The solutions associated to  $P_{1,2}$  ( $p_\mp$  in the notation in [403]) represent stiff and then decelerating solutions (actually solutions associated to a massless scalar field). For the general model including radiation we have obtained in the limit  $\phi \rightarrow \infty$ : radiation-dominated cosmological solutions; power-law scalar-field dominated inflationary cosmological solutions; matter-kinetic-potential scaling solutions and radiation-kinetic-potential scaling solutions.
11. For the model without including radiation we have proved a theorem (theorem 24) which is an extension of the theorem 4 in [403] to the STT framework. Also, we sketch the proof of the global singularity theorem 25. Theorem 25 indicates that the past asymptotic structure of non-minimally coupled scalar field theories with FRW metric, as in the FRW general relativistic case, is independent of the exact details of the potential and/or the details of the background matter and the coupling function. This is a conjecture with solid theoretical and numerical foundations (see figures 1 and 2 in 3.6.2.1). To prove that the family of solutions which asymptotically approach  $P_1$  are completely characterized by the solution space of the massless scalar field cosmological model (i.e.,  $V$  and  $\chi$  and then  $\rho$ , being dynamically insignificant in the neighborhood of the singularity  $P_1$ ) it is required to prove that this correspondence is one-to-one and continuous, which is hard to do in our scenario.
12. Using the mathematical apparatus developed in the first part of the chapter, we have investigated, for the general model including radiation, a general class of potentials containing the cases investigated in [409, 410]. In order to provide a numerical evidence for our analytical results for this class of models, we have re-examined the toy model with power-law coupling and Albrecht-Skordis potential  $V(\phi) =$

$e^{-\mu\phi}(A + (\phi - B)^2)$  investigated in [136] in presence of radiation.

13. Also we have investigated, for the general model including radiation, the important examples of higher order gravity theories  $F(R) = R + \alpha R^2$  (quadratic gravity) and  $F(R) = R^n$ . In the case of quadratic gravity we have proved in proposition 15, by an explicit computation of the center manifold, that the singular point corresponding to *de Sitter* solution is locally unstable (saddle point). This result complements the result of the proposition discussed in [405] p. 5, where it was proved the local asymptotic instability of the *de Sitter* universe for positively curved FRW models with a perfect fluid matter source and a scalar field which arises in the conformal frame of the  $R + \alpha R^2$  theory.



# Chapter 4

## Phantom dark energy with varying-mass dark matter particles

In this chapter we investigate several varying-mass dark-matter particle models in the framework of phantom cosmology. We examine whether there exist late-time cosmological solutions, corresponding to an accelerating universe and possessing dark energy and dark matter densities of the same order. Imposing exponential or power-law potentials and exponential or power-law mass dependence, we conclude that the coincidence problem cannot be solved or even alleviated. Thus, if dark energy is attributed to the phantom paradigm, varying-mass dark matter models cannot fulfill the basic requirement that led to their construction.

### 4.1 Introduction

The dynamical nature of dark energy introduces a new cosmological problem, namely why are the densities of vacuum energy and dark matter nearly equal today although they scale independently during the expansion history. The elaboration of this “coincidence” problem led to the consideration of generalized versions of the aforementioned scenarios with the inclusion of a coupling between dark energy and dark matter. Thus, various forms of “interacting” dark energy models [204, 419, 420, 421] have been constructed in order to fulfil the observational requirements. In the case of interacting quintessence one can find accelerated attractors which moreover give dark matter and dark energy density parameters of the same order, thus solving the coincidence problem [45, 193, 194, 195, 196, 197, 198, 199, 200, 422, 423], but paying the price of introducing new problems such as the justification of a non-trivial, almost tuned, sequence of cosmological epochs [424]. In interacting phantom models [189, 420, 421], the existing literature remains in some special coupling forms which suggest that the coincidence problem might be alleviated [420, 421].

An equivalent approach is to assume that dark energy and dark matter sectors interact in such a way that the dark matter particles acquire a varying mass, dependent on the scalar field which reproduces dark energy [201]. This consideration allows for a better theoretical justification, since a scalar-field-dependent varying-mass can arise from string or scalar-tensor theories [202]. Indeed, in such higher dimensional frameworks one can formulate both the appearance of the scalar field (which is related to the dilaton and moduli fields) and its effect on matter particle masses (determined by string dynamics, supersymmetry breaking, and the compactification mechanism) [203]. In quintessence scenario, such varying-mass dark matter models have been explored in cases of linear [201, 204, 203, 205], power-law [206] or exponential [193, 425, 426] scalar-field dependence. The exponential case is the most interesting since, apart from solving the coincidence problem, it allows for stable scaling behavior, that is for a large class of initial conditions the cosmological evolution converges to a common solution at late times [193, 426].

We are interested in investigating varying-mass dark matter models in scenarios where dark energy is attributed to a phantom field. Although such a framework could lead to instabilities at the quantum level [173], there have been serious attempts in overcoming these difficulties and construct a phantom theory consistent with the basic requirements of quantum field theory, with the phantom fields arising as an effective description [427]. Performing a complete phase-space analysis using various forms of mass-dependence and scalar-field potentials, we examine whether there exist stable late-time accelerating solutions which moreover solve the coincidence problem. As we will show, the coincidence problem cannot be solved in any of the investigated models. In the paper [152] we examine whether there exist late-time cosmological solutions, corresponding to an accelerating universe and possessing dark energy and dark matter densities of the same order. Imposing exponential or power-law potentials and exponential or power-law mass dependence, the coincidence problem cannot be solved or even alleviated. Thus, if dark energy is attributed to the phantom paradigm, varying-mass dark matter models cannot fulfill the basic requirement that led to their construction. In this book we improve the analysis in [152] by using the Center Manifold Theory to analyze the stability of the non-hyperbolic fixed points in the phase space of dark-matter particle models in the framework of phantom cosmology. Basically, we use these cosmological models as examples of how to apply the Center Manifold Theory in cosmology. Also, in this book we perform a Poincaré compactification process allowing to construct a global phase space containing all the cosmological information in both finite and infinite regions.



## 4.2 Phase-space analysis

In section 1.1.6 we constructed a cosmological scenario where the dark matter particles have a varying mass, depending on the phantom field. Additionally, we presented the formalism for its transformation into an autonomous dynamical system, suitable for a stability analysis (see also the analysis by one of us in [152]). In this section we introduce specific forms for  $V(\phi)$  and  $M_{DM}(\phi)$ , and we perform a complete phase-space analysis.

For the scalar field potential we consider two well studied cases of the literature, namely the exponential [193, 426]:

$$V(\phi) = V_0 e^{-\kappa\lambda_1\phi} \quad (4.1)$$

and the power-law one [206, 428]:

$$V(\phi) = V_0 \phi^{-\lambda_2}. \quad (4.2)$$

For the dark matter particle mass we consider two possible cases, namely an exponential dependence [193, 425, 426]:

$$M_{DM}(\phi) = M_0 e^{-\kappa\mu_1\phi} \quad (4.3)$$

and the power-law one [206]:

$$M_{DM}(\phi) = M_0 \phi^{-\mu_2}. \quad (4.4)$$

Therefore, in the following we consider four different models, arising from the aforementioned combinations.

In order to perform the phase-space and stability analysis of the phantom model at hand, we have to transform the aforementioned dynamical system into its autonomous form [409, 429, 430, 431]. This will be achieved by introducing the auxiliary variables:

$$\begin{aligned} x &= \frac{\kappa\dot{\phi}}{\sqrt{6}H}, \\ y &= \frac{\kappa\sqrt{V(\phi)}}{\sqrt{3}H}, \\ z &= \frac{\sqrt{6}}{\kappa\phi} \end{aligned} \quad (4.5)$$

together with  $M = \ln a$ . Thus, it is easy to see that for every quantity  $F$  we acquire

$\dot{F} = H \frac{dF}{dM}$ . Using these variables we obtain:

$$\Omega_\phi \equiv \frac{\kappa^2 \rho_\phi}{3H^2} = -x^2 + y^2, \quad (4.6)$$

$$w_\phi = \frac{-x^2 - y^2}{-x^2 + y^2}, \quad (4.7)$$

and

$$w_{tot} = -x^2 - y^2. \quad (4.8)$$

We mention that relations (4.7) and (4.8) are always valid, that is independently of the specific state of the system (they are valid in the whole phase-space and not only at the critical points). Finally, note that in the case of complete dark energy domination, that is  $\rho_{DM} \rightarrow 0$  and  $\Omega_\phi \rightarrow 1$ , we acquire  $w_{tot} \approx w_\phi \leq -1$ , as expected to happen in phantom-dominated cosmology. Finally, the deceleration parameter  $q \equiv -\frac{\ddot{a}a}{\dot{a}^2}$  is given by

$$q = \frac{1}{2} (1 - 3x^2 - 3y^2). \quad (4.9)$$

## 4.2.1 Model 1: Exponential potential and exponentially-dependent dark-matter particle mass

Inserting the auxiliary variables (4.5) into the equations of motion (1.66), (1.68), (1.69) and (1.70), we result in the following autonomous system:

$$\begin{aligned} x' &= -3x + \frac{3}{2}x(1 - x^2 - y^2) - \sqrt{\frac{3}{2}}\lambda_1 y^2 - \sqrt{\frac{3}{2}}\mu_1(1 + x^2 - y^2) \\ y' &= \frac{3}{2}y(1 - x^2 - y^2) - \sqrt{\frac{3}{2}}\lambda_1 xy. \end{aligned} \quad (4.10)$$

defined in the phase plane  $\{(x, y) | -x^2 + y^2 \leq 1, y \geq 0\}$ . Note that in this case, the auxiliary variable  $z$  is not needed.

### 4.2.1.1 Finite analysis

The critical points  $(x_c, y_c)$  of the autonomous system (4.10) are obtained by setting the left hand sides of the equations to zero. The real and physically meaningful (that is corresponding to  $y > 0$  and  $0 \leq \Omega_\phi \leq 1$ ) of them are:

$$\left( x_{c1} = -\frac{\lambda_1}{\sqrt{6}}, y_{c1} = \sqrt{1 + \frac{\lambda_1^2}{6}} \right),$$

Table 4.1: The real and physically meaningful critical points of Model 1 and their behavior.

Cr. P.	$x_c$	$y_c$	Existence	Stable for
A	$x_{c1}$	$y_{c1}$	Always	$\lambda_1 (\mu_1 - \lambda_1) < 3$
B	$x_{c2}$	$y_{c2}$	$\min\{\mu_1^2 - 3, \lambda_1^2 + 3\} \geq \lambda_1 \mu_1,$ $\mu_1 \neq \lambda_1$	Never

Table 4.2: Basic observables and conditions for acceleration for the real and physically meaningful critical points of Model 1.

Cr. P.	$\Omega_\sigma$	$w_{tot}$	Acceleration
A	1	$-\frac{1}{3}(3 + \lambda_1^2)$	Always
B	$\frac{\mu_1^2 - \lambda_1 \mu_1 - 3}{(\lambda_1 - \mu_1)^2}$	$\frac{\mu_1}{\lambda_1 - \mu_1}$	$\mu_1 < 0, \mu_1 < \lambda_1 < -2\mu_1$ $\mu_1 > 0, -2\mu_1 < \lambda_1 < \mu_1$

$$\left( x_{c2} = \frac{\sqrt{\frac{3}{2}}}{\lambda_1 - \mu_1}, y_{c2} = \frac{\sqrt{-\frac{3}{2} - \mu_1 (\lambda_1 - \mu_1)}}{|\lambda_1 - \mu_1|} \right), \quad (4.11)$$

and in table 4.1 we present the necessary conditions for their existence and their dynamical character. In table 4.2 are displayed some basic observables such that  $\Omega_\sigma$ <sup>1</sup> and  $w_{tot}$  and the conditions for acceleration for the real and physically meaningful critical points of Model 1.

Therefore, for each critical point of table 4.1, we examine the signs of the real parts of the eigenvalues of the linearization matrix, which determine the type and stability of this specific critical point. In table 4.1 we present the results of the stability analysis. In addition, in table 4.2, for each critical point we calculate the values of  $w_{tot}$  (given by relation (4.8)), and of  $\Omega_\phi$  (given by (4.6)). Thus, knowing  $w_{tot}$  we can express the acceleration condition  $w_{tot} < -1/3$  in terms of the model parameters.

The critical point A exists always and it is either a saddle point (the eigenvalues of the linearization matrix have real parts of different sign) or an attractor (the eigenvalues of the linearization matrix have negative real parts). The critical point B, if it exists, it is always a saddle point. The cosmological model at hand admits another critical point, namely C, which is unphysical since it leads to  $\Omega_\phi < 0$ . This point has coordinates  $(x_{c3} = -\sqrt{\frac{2}{3}}\mu_1, y_{c3} = 0)$  and it is either a saddle point or an attractor. If  $\mu_1(\mu_1 - \lambda_1) > 3/2$  it is an attractor and in this case, although unphysical, it can attract an open set of orbits from the interior of the physical region of the phase space.

In order to present this behavior more transparently, we evolve the autonomous system

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<sup>1</sup>In this chapter  $\Omega_\sigma$  is referred to the fractional energy density of the  $\sigma$ -field, not to the energy density of anisotropy.

(4.10) numerically for the parameters  $\lambda_1 = 0.4$  and  $\mu_1 = 2$ , and the results are shown in figure 4.1. Depending on which region of the phase-space does the system initiates, it lies

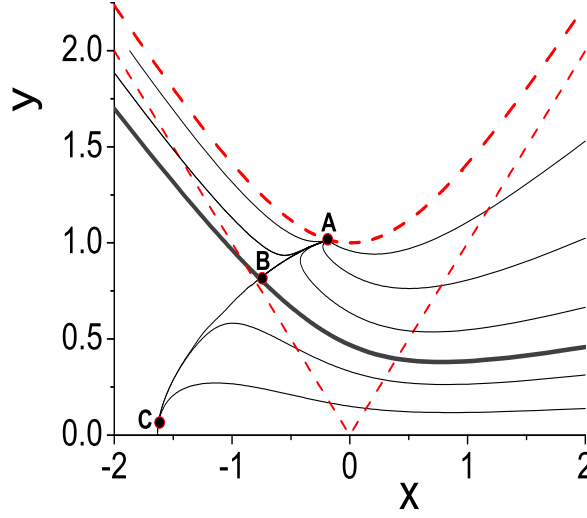


Figure 4.1: Phase plane of Model 1 for the parameter values  $\lambda_1 = 0.4$  and  $\mu_1 = 2$ . The stable manifold of B (thick curve) divides the physical part of the phase space (region corresponding to  $0 \leq \Omega_\phi \leq 1$ , bounded by the dashed (red) curves) in two regions. The orbits initially below this curve converge towards C. The orbits initially above this curve converge to A. [Taken from [152]; published with permission of Elsevier B.V.].

in the basin of attraction of either A or C, and thus it is attracted by one or the other point. In particular, the orbits initially below the stable manifold of B-points converge towards C, while the orbits initially above this curve converge to A. Interestingly, A is not the global attractor for points at the physical region (region corresponding to  $0 \leq \Omega_\phi \leq 1$ , bounded by the dashed (red) curves). However, if  $\frac{\lambda_1}{2} - \frac{\sqrt{6+\lambda_1^2}}{2} < \mu < \frac{\lambda_1}{2} + \frac{\sqrt{6+\lambda_1^2}}{2}$ , point C is always a saddle one and B does not exist. Thus, in this case A is the attractor for all the points located at the physical region. This behavior is presented in figure 4.2. Finally, for completeness we mention that in the trivial case  $\mu_1 = 0$  the origin is also a saddle point. It represents matter-dominated universe ( $\Omega_{DM} \equiv \frac{\kappa^2 \rho_{DM}}{3H^2} = 1$ ) with  $\phi$ -independent dark matter particle mass.

#### 4.2.1.2 Analysis at infinity

Owing to the fact that the dynamical system (4.10) is non-compact, there could be features in the asymptotic regime which are non-trivial for the global dynamics. Thus, in order to complete the analysis of the phase space we will now extend our study using the Poincaré central projection method.

For do that we introduce the Poincaré variables

$$x_r = \rho \cos \theta, y_r = \rho \sin \theta \quad (4.12)$$

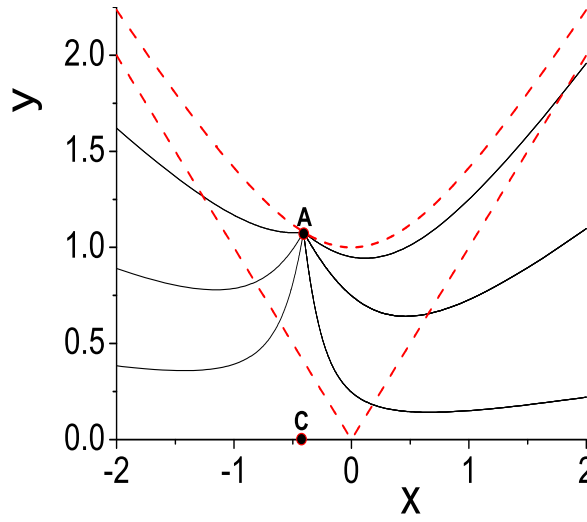


Figure 4.2: Phase plane of Model 1 for the parameter values  $\lambda_1 = 1$  and  $\mu_1 = 0.5$ . In this case the critical point B does not exist and all orbits initially at the physical region converge to A. The dashed (red) curves bound the physical part of the phase space, that is corresponding to  $0 \leq \Omega_\phi \leq 1$ . [Taken from [152]; published with permission of Elsevier B.V.].

where  $\rho = \frac{r}{\sqrt{1+r^2}}$ ,  $r = \sqrt{x^2 + y^2}$ . Thus, the points at “infinite” ( $r \rightarrow +\infty$ ) are those having  $\rho \rightarrow 1$ . Since  $y$  is required to be nonnegative,  $\phi$  varies in  $[0, \pi]$ . The region of physical interests is given by  $-\frac{\sqrt{2}}{2} \leq x_r \leq \frac{\sqrt{2}}{2}$ ,  $|x_r| \leq y_r \leq \frac{\sqrt{2}}{2}$ .

Performing the transformation (4.12), the system (4.10) becomes

$$\rho' = -\frac{3\rho^3}{2} + \frac{3}{2}(\rho^2 - 1)\cos(2\theta)\rho + \cos\theta \left( (\mu_1 - \lambda_1)\rho^2\sqrt{6 - 6\rho^2}\sin^2\theta - \sqrt{\frac{3}{2}}\mu_1\sqrt{1 - \rho^2} \right), \quad (4.13)$$

$$\theta' = 3\cos\theta\sin\theta + \left( \frac{\sqrt{6 - 6\rho^2}\mu_1}{2\rho} + \frac{\sqrt{\frac{3}{2}}(\mu_1 - \lambda_1)\rho\cos(2\theta)}{\sqrt{1 - \rho^2}} \right) \sin\theta. \quad (4.14)$$

In the limit  $\rho \rightarrow 1$ , the leading terms in (4.13)-(4.14) are

$$\rho' \rightarrow -\frac{3}{2}, \quad (4.15)$$

$$\theta' \rightarrow -\frac{\sqrt{\frac{3}{2}}(\lambda_1 - \mu_1)\cos(2\theta)\sin\theta}{\sqrt{1 - \rho^2}}. \quad (4.16)$$

The radial equation does not contain the radial coordinate, so the singular points can be obtained using the angular equation only. Setting  $\theta' = 0$ , we obtain the singular points

which are listed in table 4.3. The stability of these points is studied by analyzing first the stability of the angular coordinate and then deducing, from the sign of equation (4.15), the stability on the radial direction. Since  $\rho' < 0$ , the singular points at infinity are either saddles or sources. For simplicity we assume  $\lambda_1 \neq \mu_1$ .

Performing the above procedure we find that there is no late-time attractors in the infinite region. Thus, following the discussion in the section 4.2.1.1 the relevant late-time attractor with physical sense is the phantom-dominated super-accelerated solution,  $A$ , for the choice of parameters in the range  $\lambda_1 (\mu_1 - \lambda_1) < 3$ .

Table 4.3: Asymptotic singular points of the system (4.10) (case 1) and their stability.

Cr. P	Coordinates: $\theta, x_r, y_r$	Eigenvalue	$\rho'$	Stability
$Q_1$	$0, 1, 0$	$\begin{cases} -\infty & \text{for } \mu_1 < \lambda_1 \\ +\infty & \text{for } \mu_1 > \lambda_1 \end{cases}$	$-\frac{3}{2}$	$\begin{cases} \text{saddle} \\ \text{source} \end{cases}$
$Q_2$	$\pi, -1, 0$	$\begin{cases} +\infty & \text{for } \mu_1 < \lambda_1 \\ -\infty & \text{for } \mu_1 > \lambda_1 \end{cases}$	$-\frac{3}{2}$	$\begin{cases} \text{source} \\ \text{saddle} \end{cases}$
$Q_3$	$\frac{\pi}{4}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$	$\begin{cases} +\infty & \text{for } \mu_1 < \lambda_1 \\ -\infty & \text{for } \mu_1 > \lambda_1 \end{cases}$	$-\frac{3}{2}$	$\begin{cases} \text{source} \\ \text{saddle} \end{cases}$
$Q_4$	$\frac{3\pi}{4}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$	$\begin{cases} -\infty & \text{for } \mu_1 < \lambda_1 \\ +\infty & \text{for } \mu_1 > \lambda_1 \end{cases}$	$-\frac{3}{2}$	$\begin{cases} \text{saddle} \\ \text{source} \end{cases}$

The basic observables (4.6), (4.7), (4.8) and (4.9) are given in terms of the Poincaré variables by

$$\begin{aligned}
\Omega_\phi &\equiv \frac{(x_r - y_r)(x_r + y_r)}{x_r^2 + y_r^2 - 1} = \frac{\rho^2 \cos(2\theta)}{\rho^2 - 1}, \\
w_\phi &\equiv \frac{x_r^2 + y_r^2}{(x_r - y_r)(x_r + y_r)} = \sec(2\theta), \\
w_{\text{tot}} &\equiv \frac{x_r^2 + y_r^2}{x_r^2 + y_r^2 - 1} = \frac{\rho^2}{\rho^2 - 1}, \\
q &\equiv \frac{4x_r^2 + 4y_r^2 - 1}{2(x_r^2 + y_r^2 - 1)} = \frac{1 - 4\rho^2}{2 - 2\rho^2}.
\end{aligned} \tag{4.17}$$

Taking the limit  $\rho \rightarrow 1^-$  in the expressions (4.17), it is easy to see that  $q \rightarrow -\infty$  and  $w_{\text{tot}} \rightarrow -\infty$ . That is, the points at infinity represents supper-accelerating ( $q \ll 0$ ) phantom solutions ( $w_{\text{tot}} \ll -1$ ), which can be physical or unphysical depending whether or not  $0 \leq \Omega_\phi \leq 1$ , or, equivalently, whether or not  $-\frac{\sqrt{2}}{2} \leq x_r \leq \frac{\sqrt{2}}{2}, |x_r| \leq y_r \leq \frac{\sqrt{2}}{2}$ . Thus the solutions  $Q_1$  and  $Q_2$  are unphysical, with  $\Omega_\phi \rightarrow -\infty$ . At the singular point  $Q_3$  we have that  $\Omega_\phi = 0, w_\phi = -\infty$  and at the singular point  $Q_4$  we have  $\Omega_\phi = 0, w_\phi = +\infty$ . Although they are matter dominated solutions, since  $q \rightarrow -\infty$  and  $w_{\text{tot}} \rightarrow -\infty$ , they

mimics phantom behavior. In fact they correspond to big-rip singularities.

The system (4.13)-(4.14) has an apparent singularity in  $\rho = 0, \sin \theta = 0$ , which is due to the spherical coordinate system. Thus, for numerical examinations it is more convenient to use the cartesian coordinates  $x_r, y_r$ . The system reads

$$\begin{aligned} x'_r &= \frac{1}{2} \left( \frac{\sqrt{6}((2x_r^2-1)y_r^2\lambda_1 - (x_r^2-1)(2y_r^2-1)\mu_1)}{\sqrt{1-x_r^2-y_r^2}} - 3x_r(2y_r^2+1) \right), \\ y'_r &= -\frac{1}{2}y_r(2y_r^2-1) \left( \frac{\sqrt{6}x_r(\mu_1-\lambda_1)}{\sqrt{1-x_r^2-y_r^2}} + 3 \right). \end{aligned} \quad (4.18)$$

In figure 4.3 we present the Poincaré (global) phase plane of Model 1 for the parameter values  $\lambda_1 = 0.4$  and  $\mu_1 = 2$ . There are two attractors in the finite region: a physical one A, and an unphysical state C. The orbits initially above the stable manifold of B converge to A. The points at infinity  $Q_1$  and  $Q_4$  are sources, whereas  $Q_2$  and  $Q_3$  are saddles.

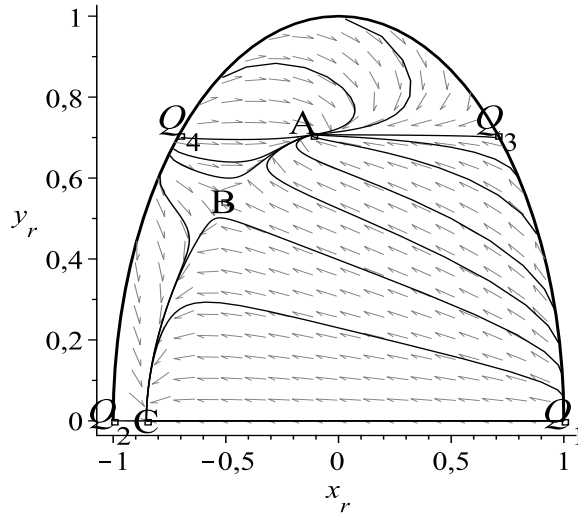


Figure 4.3: Poincaré (global) phase plane of Model 1 for the parameter values  $\lambda_1 = 0.4$  and  $\mu_1 = 2$ . The attractors in the finite region are A which is physical ( $0 \leq \Omega_\phi \leq 1$ ) and C. The orbits initially above the stable manifold of B converge to A (see figure 4.1). The points at infinity  $Q_1$  and  $Q_4$  are sources, whereas  $Q_2$  and  $Q_3$  are saddles.

In the figure 4.4 we show the Poincaré (global) phase plane of Model 1 for the parameter values  $\lambda_1 = 1.0$  and  $\mu_1 = 0.5$ . The points at infinity  $Q_2$  and  $Q_3$  are sources, whereas  $Q_1$  and  $Q_4$  are saddles. Thus, the scalar field dominated solution A is the global attractor for both finite and infinite regions for the choice of parameters in the range  $\lambda_1(\mu_1 - \lambda_1) < 3$ .

#### 4.2.1.3 Cosmological implications and discussion: Model 1

Having performed a complete phase-space analysis we can discuss the corresponding cosmological behavior. A general remark is that this behavior is radically different from

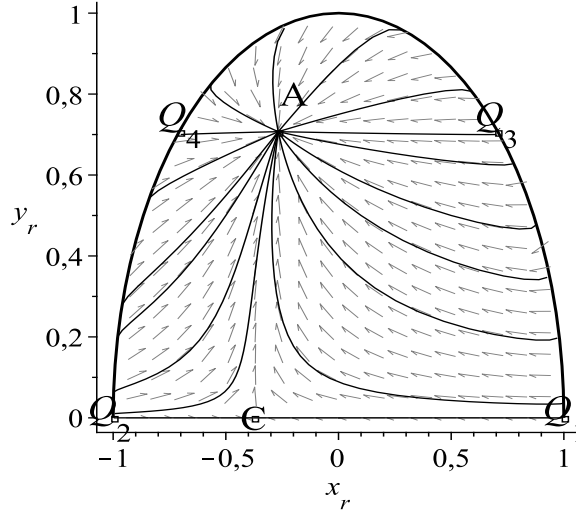


Figure 4.4: Poincaré (global) phase plane of Model 1 for the parameter values  $\lambda_1 = 1.0$  and  $\mu_1 = 0.5$ . The attractor in the finite region is the singular point A which is physical ( $0 \leq \Omega_\phi \leq 1$ ) (see figure 4.2). The unphysical state C is a saddle. The points at infinity  $Q_2$  and  $Q_3$  are sources, whereas  $Q_1$  and  $Q_4$  are saddles.

the corresponding quintessence scenarios with the same potentials and mass-functions [193, 201, 203, 204, 205, 206, 425, 426]. Additionally, a common feature of almost all the phantom models previously studied is the existence of attractors with  $w_\phi \leq -1$  in the whole phase-space [189, 432], and thus, independently of the specific scenario and of the imposed initial conditions, the universe always lies below the phantom divide, as it is expected for phantom cosmology. This global behavior is not always realized in the case of exponentially dependent dark-matter mass, and additional constraints must be imposed.

Apart from acquiring acceleration, in this work we examine whether the above constructed varying dark-matter-mass models can solve or alleviate the coincidence problem. Thus, assuming as usual that the present universe is already at a late-time attractor, we calculate  $\Omega_\phi$  in all stable fixed points, and if  $0 < \Omega_\phi < 1$  then the coincidence problem is solved since  $\Omega_\phi$  and  $\Omega_{DM}$  will be of the same order of magnitude as suggested by observations. On the contrary,  $\Omega_\phi = 1$  corresponds to a universe completely dominated by dark energy, while  $\Omega_\phi = 0$  (that is  $\Omega_{DM} = 1$ ) to one completely dominated by dark matter, both in contrast with observations.

Finally, we mention that as long as the interaction responsible for the varying dark-matter particle mass is not too strong, the standard cosmology can be always recovered. On the other hand, since we assume that the universe is currently at an attractor, its state is independent of the initial conditions. Thus, we can switch on the interaction and consider as initial conditions the end of the known epochs of standard Big Bang cosmology, in order to avoid disastrous interference.



For the model 1, the critical point B is unstable, and therefore it cannot be a late-time cosmological solution. The only relevant critical point is A, which is a stable fixed point for  $\lambda_1 (\mu_1 - \lambda_1) < 3$ . As can be seen from table 4.1, it corresponds to an accelerating universe with  $\Omega_\phi = 1$ , that is to complete dark-energy domination. Thus, this specific cosmological solution cannot solve the coincidence problem. Furthermore, the fact that  $w_{tot}$  is not only less than  $-1/3$ , as required by the acceleration condition, but it is always less than  $-1$ , leads to  $\dot{H} > 0$  at all times. Therefore, this solution corresponds to a super-accelerating universe [433], that is with a permanently increasing  $H$ , resulting to a Big Rip. This behavior is common in phantom cosmology [99, 432]. Another interesting feature of exponential potential and exponentially-dependent dark-matter particle mass phantom model is the existence of past big-rip singularities at infinity (either  $Q_{1,4}$  or  $Q_{2,3}$ ).

A remarkable feature of this model, as well as of Model 3, is that if there exist scaling solutions, then, for a wide region of the parameter space, the stable manifold of the corresponding critical point marks the basin of attraction of either a phantom attractor or an unphysical attracting state. Thus, there exist an open set of orbits of the physical region that converge to an unphysical state instead to a phantom solution. This behavior was revealed analytically and it was confirmed through numerical elaboration, and seems to be typical in the case of exponentially-dependent dark-matter mass in the phantom framework. To avoid dealing with unphysical states, we can either restrict the physical portion of the phase-space to the region above the stable manifold of the scaling solutions, or restrict the parameter-space itself. In both cases we obtain an additional constraint, that was not present in previous studies of phantom cosmology [189, 420, 421], which further weakens the applicability of the model.

In summary, Model 1, that is an exponential potential and an exponentially-dependent dark-matter particle mass, cannot act as a candidate for solving the coincidence problem.

#### 4.2.2 Model 2: Power-law potential and power-law-dependent dark-matter particle mass

Inserting the auxiliary variables (4.5) into the equations of motion (1.66), (1.68), (1.69) and (1.70), we result in the following autonomous system:

$$\begin{aligned} x' &= -3x + \frac{3}{2}x(1 - x^2 - y^2) - \frac{\lambda_2 y^2 z}{2} - \frac{\mu_2}{2}z(1 + x^2 - y^2) \\ y' &= \frac{3}{2}y(1 - x^2 - y^2) - \frac{\lambda_2 x y z}{2} \\ z' &= -x z^2. \end{aligned} \tag{4.19}$$

Table 4.4: The real and physically meaningful critical points of Model 2 and their behavior.

Cr. P.	$x_c$	$y_c$	$z_c$	Existence	Stable manifold	$\Omega_\sigma$	$w_{tot}$	Acceleration
D	$x_{c4}$	$y_{c4}$	$z_{c4}$	Always	1-D	0	0	Never
E	$x_{c5}$	$y_{c5}$	$z_{c5}$	Always	2-D	1	-1	Always

#### 4.2.2.1 Finite analysis

The real and physically meaningful critical points are

$$\begin{aligned} (x_{c4} = 0, y_{c4} = 0, z_{c4} = 0), \\ (x_{c5} = 0, y_{c5} = 1, z_{c5} = 0), \end{aligned} \quad (4.20)$$

and in table 4.4 we present the necessary conditions for their existence.

In this case, the critical points are non-hyperbolic, that is there exists always at least a zero eigenvalue. We mention that for non-hyperbolic critical points the result of linearization cannot be applied in order to investigate the local stability of the system (the system can be unstable to small perturbations on the initial condition or to small perturbations on the parameters) [359, 358, 418]. However, it is possible to get information about the existence and the dimensionality of the stable manifold by applying the center manifold theorem [359]. Doing so we deduce that the dimensionality of the local stable manifold is 1 and 2 for D and E respectively. In particular, the stable manifold of D is tangent, at the critical point, to the  $x$ -axis, while the stable manifold of E is tangent, at the critical point, to the  $xy$ -plane. The existence of an 1D stable manifold for D, implies that the orbits asymptotic to D as  $t \rightarrow -\infty$  are contained in either an unstable or center manifold (each one of dimensionality 1, that is a curve). There are some exceptional orbits converging to D as  $t \rightarrow +\infty$ , but these have a zero measure. On the other hand, the fact that E has a 2D stable manifold implies that there exists a non-zero-measure set of orbits that converges to E as  $t \rightarrow +\infty$ . Finally, there are some exceptional orbits contained in its center manifold that cannot be classified by linearization.

#### 4.2.2.2 Stability of de Sitter solution for Power-law potential and power-law-dependent dark-matter particle mass.

The singular point  $E$  represents the de Sitter solution for Power-law potential and power-law-dependent dark-matter particle mass. In this section we will analyze the stability of the center manifold of  $E$  for the vector field (4.19).

**Proposition 16** *For  $\lambda_2 < 0$ , the singular point  $E : (x_{c5} = 0, y_{c5} = 1, z_{c5} = 0)$  of the system (4.19) is locally asymptotically stable. For  $\lambda_2 > 0$ ,  $E$  is locally unstable (saddle*

type). For  $\lambda_2 = 0$ ,  $E$  is stable but not asymptotically stable.

In order to determine the local center manifold of (4.19) at the origin we have to transform the system into a form suitable for the application of the center manifold theorem (see section 2.2.5.3 for a summary of the techniques involved in the proof).

**Proof.**

**Case 1.** Let us assume that  $\lambda_2 \neq 0$ .

In order to translate  $E$  to the origin and transforming the linear part of the resulting vector field into its Jordan canonical form, we define new variables  $(u, v_1, v_2) \equiv \mathbf{x}$ , by the equations

$$u = -\frac{z\lambda_2}{6}, \quad v_1 = y - 1, \quad v_2 = x + \frac{z\lambda_2}{6}$$

so that

$$\begin{pmatrix} u' \\ v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} f(u, v_1, v_2) \\ g_1(u, v_1, v_2) \\ g_2(u, v_1, v_2) \end{pmatrix} \quad (4.21)$$

where

$$\begin{aligned} f(u, v_1, v_2) &= \frac{6u^2(u+v_2)}{\lambda_2}, \quad g_1(u, v_1, v_2) = \frac{3}{2}(v_1 + 1)(u^2 - v_2^2) - \\ &\frac{3}{2}v_1^2(v_1 + 3), \quad \text{and} \quad g_2(u, v_1, v_2) = -\frac{3(\lambda_2 - 2\mu_2 + 4)u^3}{2\lambda_2} - \frac{3v_2(3\lambda_2 - 4\mu_2 + 4)u^2}{2\lambda_2} + \\ &\left(\left(\frac{3}{2} - \frac{3\mu_2}{\lambda_2}\right)v_1^2 + \left(3 - \frac{6\mu_2}{\lambda_2}\right)v_1 + v_2^2\left(\frac{3\mu_2}{\lambda_2} - \frac{9}{2}\right)\right)u - \frac{3v_2^3}{2} - \frac{3v_1^2v_2}{2} - 3v_1v_2. \end{aligned}$$

The system (4.21) is written in diagonal form

$$\begin{aligned} u' &= Cu + f(u, \mathbf{v}) \\ \mathbf{v}' &= P\mathbf{v} + \mathbf{g}(u, \mathbf{v}), \end{aligned} \quad (4.22)$$

where  $(u, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^2$ ,  $C$  is the zero  $1 \times 1$  matrix,  $P$  is a  $2 \times 2$  matrix with negative eigenvalues and  $f, \mathbf{g}$  vanish at  $\mathbf{0}$  and have vanishing derivatives at  $\mathbf{0}$ . The center manifold theorem 13 asserts that there exists a 1-dimensional invariant local center manifold  $W^c(\mathbf{0})$  of (4.22) tangent to the center subspace (the  $\mathbf{v} = \mathbf{0}$  space) at  $\mathbf{0}$ . Moreover,  $W^c(\mathbf{0})$  can be represented as

$$W^c(\mathbf{0}) = \{(u, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^2 : \mathbf{v} = \mathbf{h}(u), |u| < \delta\}; \quad \mathbf{h}(0) = \mathbf{0}, D\mathbf{h}(0) = \mathbf{0}$$

for  $\delta$  sufficiently small (see definition 13). The restriction of (4.22) to the center manifold is (see definition 2.36)

$$u' = f(u, \mathbf{h}(u)). \quad (4.23)$$

According to Theorem 14, if the origin  $u = 0$  of (4.23) is stable (asymptotically stable) (unstable) then the origin of (4.22) is also stable (asymptotically stable) (unstable). Therefore, we have to find the local center manifold, i.e., the problem reduces to the computation of  $\mathbf{h}(u)$ .

Substituting  $\mathbf{v} = \mathbf{h}(u)$  in the second component of (4.22) and using the chain rule,  $\mathbf{v}' = D\mathbf{h}(u)u'$ , one can show that the function  $\mathbf{h}(u)$  that defines the local center manifold satisfies

$$D\mathbf{h}(u)[f(u, \mathbf{h}(u))] - P\mathbf{h}(u) - \mathbf{g}(u, \mathbf{h}(u)) = 0. \quad (4.24)$$

According to Theorem 15, equation (4.24) can be solved approximately by using an approximation of  $\mathbf{h}(u)$  by a Taylor series at  $u = 0$ . Since  $\mathbf{h}(0) = \mathbf{0}$  and  $D\mathbf{h}(0) = \mathbf{0}$ , it is obvious that  $\mathbf{h}(u)$  commences with quadratic terms. We substitute

$$\mathbf{h}(u) =: \begin{bmatrix} h_1(u) \\ h_2(u) \end{bmatrix} = \begin{bmatrix} a_1u^2 + a_2u^3 + O(u^4) \\ b_1u^2 + b_2u^3 + O(u^4) \end{bmatrix}$$

into (4.24) and set the coefficients of like powers of  $u$  equal to zero to find the unknowns  $a_1, b_1, \dots$

We find that the non-zero coefficients are

$$a_2 = \frac{1}{2}, \quad b_3 = -\frac{2}{\lambda_2},$$

Therefore, (4.23) yields

$$u' = \frac{6u^3}{\lambda_2} - \frac{12u^5}{\lambda_2^2} + O(u^6). \quad (4.25)$$

Neglecting the error terms, this is a gradient-like equation (i.e.,  $u' = -\nabla U(u)$ ) with potential  $U(u) = \frac{2u^6}{\lambda_2^2} - \frac{3u^4}{2\lambda_2}$  for which the origin is a degenerate minimum provided  $\lambda_2 < 0$  and a degenerated maximum provided  $\lambda_2 > 0$ . Thus, for  $\lambda_2 < 0$ , the origin  $u = 0$  of (4.25) is locally asymptotically stable. Hence, the origin  $\mathbf{u} = \mathbf{0}$  of the full three-dimensional system is asymptotically stable. For  $\lambda_2 > 0$  the origin is locally unstable (saddle type)

**Case 2.** Let us assume that  $\lambda_2 = 0$ . In this case in order to translate  $E$  to the origin and reducing the linear part of the vector field to its Jordan canonical form, we define new variables the  $u = z$ ,  $v_1 = x$ ,  $v_2 = y - 1$ . Thus, the system (4.19) reduces to

so that

$$\begin{pmatrix} u' \\ v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} -u^2v_1 \\ \frac{1}{2}(-3v_1^3 - u\mu_2v_1^2 - 3v_2(v_2+2)v_1 + uv_2(v_2+2)\mu_2) \\ -\frac{3}{2}((v_2+1)v_1^2 + v_2^2(v_2+3)) \end{pmatrix} \quad (4.26)$$

The equations for the center manifold of the origin (4.24) reduces to

$$\begin{aligned} u\mu_2h_2(h_2+2) &= h_1(-2h_1'u^2 + \mu_2h_1u + 3h_1^2 + 3h_2(h_2+2) + 6), \\ 3(h_2+1)(h_1^2 + h_2(h_2+2)) &= 2u^2h_1h_2'. \end{aligned} \quad (4.27)$$

We obtain, using a Taylor series at  $u = 0$ , that the solution of (4.27) satisfying  $\mathbf{h}(0) = 0$ ,  $D\mathbf{h}(0) = \mathbf{0}$  is the trivial solution to arbitrary order. This means that the center manifold of  $E$  is a small segment contained in the  $z$ -axis.

In order to examine the stability of the origin for the flow of (4.26) we proceed as follows. Using spherical coordinates

$$u = r \cos \varphi \sin \theta, v_1 = r \sin \theta \sin \varphi, v_2 = r \cos \theta \quad (4.28)$$

and taking the limit  $r \rightarrow 0$  the angular equations  $\theta', \varphi'$  becomes

$$\varphi' \rightarrow -3 \cos \varphi \sin \varphi, \theta' \rightarrow 3 \cos \theta \cos^2 \varphi \sin \theta. \quad (4.29)$$

Solving the approximate equations (4.29) we obtain

$$\theta(\tau) = \tan^{-1} \left( e^{2c_2} \sqrt{e^{6\tau} + e^{4c_1}} \right), \varphi(\tau) = \tan^{-1} \left( e^{2c_1-3\tau} \right), \quad (4.30)$$

where  $c_1$  and  $c_2$  are integration constants.

By Taylor expanding the radial equation around  $r = 0$  we obtain the equation

$$r' = -\frac{3}{4} (-2 \cos(2\varphi) \sin^2 \theta + \cos(2\theta) + 3) r + O(r^2). \quad (4.31)$$

By substituting the first order solution (4.30) into the equation (4.31) and solving the resulting equation we obtain

$$r(\tau) = e^{-3\tau} \sqrt{1 + e^{4(c_1+c_2)} + e^{6\tau+4c_2}c_3}, \quad (4.32)$$

where  $c_3$  is an integration constant. Substituting (4.30) and (4.32) in (4.28) and taking the limit as  $\tau \rightarrow +\infty$  we obtain  $u \rightarrow u_0, v_1 \rightarrow 0, v_2 \rightarrow 0$  where  $u_0 = u(0)$ . Let be  $\epsilon > 0$  an arbitrary number. Then there exists a  $\delta > 0$ , such that  $\delta < \epsilon$ . Let us consider the solution with initial value  $u(0) = u_0, v_1(0) = v_{10}, v_2(0) = v_{20}$ , with  $u_0^2 + v_{10}^2 + v_{20}^2 < \delta^2$ . Since  $u \rightarrow u_0$ , satisfying  $|u_0| < \delta$ , then the solution,  $\mathbf{x}(\tau, \mathbf{x}_0)$  passing through  $\mathbf{x}_0 = (u_0, v_{10}, v_{20})$  at  $\tau = 0$ , satisfies  $\|\mathbf{x}(\tau, \mathbf{x}_0)\| < \epsilon$ , for  $\tau$  arbitrarily large. In this way we prove the stability (but not asymptotic stability) of the  $E$ . ■

In summary, we indeed find that the center manifold of  $E$  attracts an open set of orbits provided  $\lambda_2 \leq 0$ . On the other hand, if  $\lambda_2 > 0$  the orbits located near the center manifold of  $E$  blow up in a finite time. This point does not allow for a solution of the coincidence

problem (it always possesses  $\Omega_\phi = 1$ ).

Numerical investigation reveals the above features. In fig. 4.5 we depict orbits projected in the  $xy$ -plane, as they arise from numerical evolution in the case of  $\lambda_2 = -0.5$  and  $\mu_2 = 0.5$ .

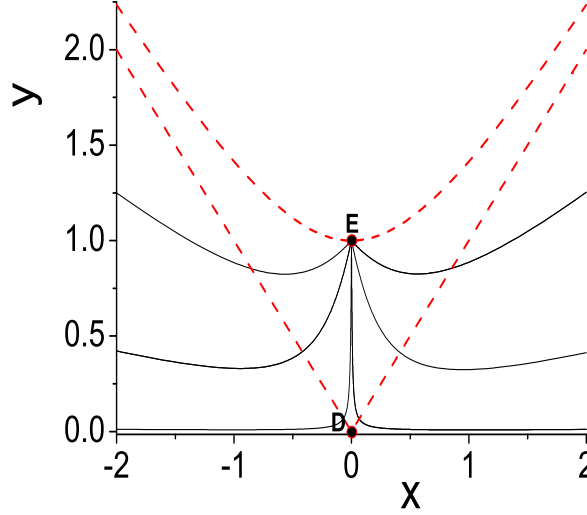


Figure 4.5:  $xy$ -projection of the phase-space of Model 2, for the parameter values  $\lambda_2 = -0.5$  and  $\mu_2 = 0.5$ . The critical point E (representing de Sitter solutions) is the attractor of the system. The dashed (red) curves bound the physical part of the phase space, that is corresponding to  $0 \leq \Omega_\phi \leq 1$ . [Taken from [152]; published with permission of Elsevier B.V.].

#### 4.2.2.3 Analysis at infinity

Introducing the Poincaré variables

$$x_r = \rho \cos \theta \sin \psi, y_r = \rho \sin \theta \sin \psi, z_r = \rho \cos \psi \quad (4.33)$$

where  $\rho = \frac{r}{\sqrt{1+r^2}}$ ,  $r = \sqrt{x^2 + y^2 + z^2}$ . Thus, the points at “infinite” ( $r \rightarrow +\infty$ ) are those having  $\rho \rightarrow 1$ . Since  $y, z$  are required to be nonnegative,  $\theta$  varies in  $[0, \pi]$ , and  $\psi$  varies in  $[0, \frac{\pi}{2}]$ .

Performing the transformation (4.33), the system (4.19) becomes

$$\begin{aligned}
\rho' = & -\frac{3}{2}\rho^3 \sin^4 \psi + \frac{1}{4}(\lambda_2 - \mu_2)\rho^3 \cos(3\theta) \cos \psi \sin^3 \psi + \\
& + \frac{3}{2}\rho (\rho^2 - 1) \cos(2\theta) \sin^2 \psi + \\
& - \frac{1}{16}\rho ((\lambda_2 - 3\mu_2 + 4)\rho^2 + 4\mu_2) \cos \theta \sin(2\psi) + \\
& + \frac{1}{32}(\lambda_2 + \mu_2 - 4)\rho^3 \cos \theta \sin(4\psi), \tag{4.34}
\end{aligned}$$

$$\begin{aligned}
\theta' = & \frac{(\lambda_2 - \mu_2)(\sin(3\theta) - \sin \theta) \sin(2\psi)\rho^2}{8(\rho^2 - 1)} + \\
& + \frac{1}{2}(6 \cos \theta + \mu_2 \cot \psi) \sin \theta, \tag{4.35}
\end{aligned}$$

$$\begin{aligned}
\psi' = & -\frac{1}{2} \cos \psi \left( 3 \sin \psi \left( \frac{\rho^2 \sin^2 \psi}{1 - \rho^2} + \cos(2\theta) \right) + \right. \\
& \left. + \cos \theta \cos \psi \left( \frac{\rho^2(\lambda_2 + (\mu_2 - \lambda_2) \cos(2\theta) - 2) \sin^2 \psi}{1 - \rho^2} + \mu_2 \right) \right). \tag{4.36}
\end{aligned}$$

In the limit  $\rho \rightarrow 1$ , the leading terms in (4.34)-(4.36) are

$$\begin{aligned}
\rho' \rightarrow & -\frac{3}{2} \sin^4 \psi + \frac{1}{4}(\lambda_2 - \mu_2) \cos(3\theta) \cos \psi \sin^3(\psi) + \\
& - \frac{1}{16}(\lambda_2 + \mu_2 + 4) \cos \theta \sin(2\psi) + \frac{1}{32}(\lambda_2 + \mu_2 - 4) \cos \theta \sin(4\psi), \tag{4.37}
\end{aligned}$$

$$\theta' \rightarrow \frac{(\lambda_2 - \mu_2)(\sin \theta - \sin(3\theta)) \sin(2\psi)}{8(1 - \rho^2)}, \tag{4.38}$$

$$\psi' \rightarrow \frac{\cos \psi (\cos \theta (-\lambda_2 + (\lambda_2 - \mu_2) \cos(2\theta) + 2) \cos \psi - 3 \sin \psi) \sin^2 \psi}{2(1 - \rho^2)}. \tag{4.39}$$

The radial equation does not contain the radial coordinate, so the singular points can be obtained using the angular equations only. Setting  $\theta' = 0, \psi' = 0$ , we obtain the singular points which are listed in table 4.5. The stability of these points is studied by analyzing first the stability of the angular coordinates and then deducing, from the sign of equation (4.37), the stability on the radial direction.

Table 4.5: Asymptotic singular points of the system (4.19) (case 2) and their stability. We use the notations  $\alpha = \frac{3\sqrt{2}}{\sqrt{22-4\lambda_2+\lambda_2^2}}$  and  $\beta = \frac{3}{\sqrt{13-4\mu_2+\mu_2^2}}$ ,  $\epsilon = \text{sign}(\lambda_2 - \mu_2)$ ,  $\delta = \text{sign}(-26 + 4\lambda_2 + \lambda_2^2)$ ,  $\eta = \text{sign}(-11 - 4\mu_2 + \mu_2^2)$ , and  $\mu_- = \frac{1}{3} \left( 5 - \sqrt[3]{\frac{2}{245-9\sqrt{741}}} - \sqrt[3]{\frac{1}{2} (245 - 9\sqrt{741})} \right) \approx -0.47$ . NH stands for nonhyperbolic.

Cr. P	Coordinates $\theta, \psi, x_r, y_r, z_r$	Eigenvalues	$\rho'$	Stability
$Q_5$	$0, 0, 0, 0, 1$	$0, 0$	$0$	NH; 3D center manifold
$Q_6$	$0, \frac{\pi}{2}, 1, 0, 0$	$0, +\infty$	$-\frac{3}{2}$	NH; 2D unstable manifold
$Q_7$	$\pi, \frac{\pi}{2}, -1, 0, 0$	$0, +\infty$	$-\frac{3}{2}$	NH; 2D unstable manifold
$Q_8$	$\frac{\pi}{4}, \frac{\pi}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0$	$0, +\infty$	$-\frac{3}{2}$	NH; 2D unstable manifold
$Q_9$	$\frac{3\pi}{4}, \frac{\pi}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0$	$0, +\infty$	$-\frac{3}{2}$	NH; 2D unstable manifold
$Q_{10}$	$\frac{\pi}{4}, \cos^{-1}(\alpha), \frac{\alpha \lambda_2-2 }{6}, \frac{\alpha \lambda_2-2 }{6}, \alpha$	$\epsilon\infty, \delta\infty$ if $\lambda_2 > 2$ $\epsilon\infty, -\infty$ if $\lambda_2 < 2$	$< 0$	source if $\lambda_2 > 2 + \sqrt{30}$ and $\mu_2 < \lambda_2$ saddle, otherwise
$Q_{11}$	$\frac{3\pi}{4}, \cos^{-1}(\alpha), -\frac{\alpha \lambda_2-2 }{6}, \frac{\alpha \lambda_2-2 }{6}, \alpha$	$-\epsilon\infty, -\infty$ if $\lambda_2 > 2$ $-\epsilon\infty, \delta\infty$ if $\lambda_2 < 2$	$< 0$ if $\lambda_2 < -1$ $> 0$ if $\lambda_2 > -1$ , and $\lambda_2 \neq 2$	sink, if $\lambda_2 > -1$ , and $\lambda_2 \neq 2$ , and $\mu_2 < \lambda_2$ saddle, otherwise
$Q_{12}$	$0, \cos^{-1}(\beta), \frac{\beta \mu_2-2 }{3}, 0, \beta$	$-\epsilon\infty, -\infty$ if $\mu_2 < 2$ $-\epsilon\infty, \eta\infty$ if $\mu_2 > 2$	$< 0$	source, if $\mu_2 > 2 + \sqrt{15}$ , and $\lambda_2 < \mu_2$ , saddle, otherwise
$Q_{13}$	$\pi, \cos^{-1}(\beta), -\frac{\beta \mu_2-2 }{3}, 0, \beta$	$\epsilon\infty, \eta\infty$ if $\mu_2 < 2$ $\epsilon\infty, -\infty$ if $\mu_2 > 2$	$< 0$ if $\mu_2 < \mu_-$ $> 0$ if $\mu_2 > \mu_-, \mu \neq 2$	source, if $\mu_2 < \lambda_2 \leq 2 - \sqrt{15}$ , or $\mu_2 < 2 - \sqrt{15} < \lambda_2$ sink, if $\lambda_2 < \mu_2, \mu_2 > 2$ or $\lambda_2 \leq \mu_- < \mu_2 < 2$ or $\mu_- < \lambda_2 < \mu_2 < 2$ ; saddle, otherwise



In order to perform the numerical experiments for the system (4.34)-(4.36) it is useful to rewrite the system in the cartesian coordinates  $x_r, y_r, z_r$ . The system (4.34)-(4.36) becomes

$$\begin{aligned}
x'_r &= \frac{1}{2} \left( (\mu_2 - (\mu_2 - 2)x_r^2) z_r^3 + 3x_r (-x_r^2 + y_r^2 + 1) z_r^2 + \right. \\
&\quad \left. - (\lambda_2 (1 - 2x_r^2) y_r^2 + \mu_2 (x_r^2 - 1) (2y_r^2 - 1)) z_r \right. \\
&\quad \left. + 3x_r (x_r^2 + y_r^2 - 1) (2y_r^2 + 1) \right), \\
y'_r &= \frac{1}{2} y_r \left( -(\mu_2 - 2)x_r z_r^3 - 3(x_r^2 - y_r^2 + 1) z_r^2 + \right. \\
&\quad \left. + (\lambda_2 - \mu_2)x_r (2y_r^2 - 1) z_r + \right. \\
&\quad \left. + 3(x_r^2 + y_r^2 - 1) (2y_r^2 - 1) \right), \\
z'_r &= \frac{1}{2} z_r \left( (6y_r^2 - 3z_r^2 + 3) x_r^2 + \right. \\
&\quad \left. + z_r (2(\lambda_2 - \mu_2)y_r^2 - (\mu_2 - 2)z_r^2 + \mu_2 - 2) x_r + \right. \\
&\quad \left. + 3y_r^2 (2y_r^2 + z_r^2 - 1) \right)
\end{aligned} \tag{4.40}$$

where we have used the time re-scaling

$$d\tau \rightarrow \frac{d\tau}{1 - \rho^2}$$

which leave invariant the orbits of the phase-space and the time direction (see theorem 4).

By an explicit calculation we find that that center manifolds of  $Q_6$  and  $Q_8$  for the flow of (4.40) is the arc  $x_r = \sqrt{1 - y_r^2}$  and that the center manifolds of  $Q_7$  and  $Q_9$  is the arc  $x_r = -\sqrt{1 - y_r^2}$ . These center manifolds are unstable. Thus, the singular points  $Q_{6,7,8,9}$  are local sources.

To examine the stability of  $Q_5$  we use Normal forms calculations. Let us assume  $\mu_2 \neq 0$ .

First, by introducing the linear change of coordinates

$$u_1 = \frac{x_r}{\mu_2}, u_2 = z_r - 1, u_3 = y_r$$

the system around  $Q_5$  becomes

$$\begin{aligned}
u'_1 &= \mu_2 u_1^2 + u_2 + \frac{3}{2} u_2 (2u_1 + u_2) + u_3^2 \left( 1 - \frac{\lambda_2}{2\mu_2} \right) + \mathcal{O}(3), \\
u'_2 &= -u_1 u_2 (\mu_2 - 2) \mu_2 + \mathcal{O}(3), \\
u'_3 &= -\frac{1}{2} u_3 (6u_2 + u_1 (\lambda_2 - 2) \mu_2) + \mathcal{O}(3)
\end{aligned} \tag{4.41}$$

Observe that the phase space is compact since

$$u_1 \in \left[ -\frac{1}{|\mu_2|}, \frac{1}{|\mu_2|} \right], u_2 \in [-1, 1], u_3 \in [0, 1].$$

The linear part of the vector field (4.41) is given by

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us consider the linear operator  $\mathbf{L}_J^{(2)}$  associated to  $J$  that assigns to  $\mathbf{h}(\mathbf{u}) \in H^2$  the Lie bracket of the vector fields  $\mathbf{A}\mathbf{u}$  and  $\mathbf{h}(\mathbf{u})$ :

$$\begin{aligned} \mathbf{L}_J^{(2)} : H^2 &\rightarrow H^2 \\ \mathbf{h} &\rightarrow \mathbf{L}_J \mathbf{h}(\mathbf{u}) = \mathbf{D}\mathbf{h}(\mathbf{u})\mathbf{J}\mathbf{u} - \mathbf{J}\mathbf{h}(\mathbf{u}). \end{aligned} \quad (4.42)$$

where  $H^2$  the real vector space of vector fields whose components are homogeneous polynomials of degree 2. The canonical basis for the real vector space of 3-dimensional vector fields whose components are homogeneous polynomials of degree 2 is given by

$$\begin{aligned} H^2 = & \text{span} \left\{ \begin{pmatrix} u_1^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 u_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 u_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2 u_3 \\ 0 \\ 0 \end{pmatrix}, \right. \\ & \begin{pmatrix} u_3^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1 u_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1 u_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2^2 \\ 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ u_2 u_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_3^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1 u_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1 u_3 \end{pmatrix}, \\ & \left. \begin{pmatrix} 0 \\ 0 \\ u_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_2 u_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_3^2 \end{pmatrix} \right\} \end{aligned} \quad (4.43)$$

By computing the action of  $\mathbf{L}_J^{(2)}$  on each basis element on  $H^2$  we have

$$\begin{aligned}
\mathbf{L}_J^{(2)}(H^2) = \\
\text{span} \left\{ \begin{pmatrix} u_1^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 u_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 u_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2 u_3 \\ 0 \\ 0 \end{pmatrix}, \right. \\
\begin{pmatrix} u_3^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1 u_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2 u_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1 u_2 \end{pmatrix}, \\
\left. \begin{pmatrix} 0 \\ 0 \\ u_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_2 u_3 \end{pmatrix} \right\}. \tag{4.44}
\end{aligned}$$

Thus, the second order terms that are linear combinations of the six vectors in (4.44) can be eliminated [358]. To determine the nature of the second order terms that cannot be eliminated we must compute the complementary space of (4.44) which is

$$\begin{aligned}
G^2 = \\
\text{span} \left\{ \begin{pmatrix} 0 \\ u_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1 u_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_3^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1 u_3 \end{pmatrix}, \right. \\
\left. \begin{pmatrix} 0 \\ 0 \\ u_3^2 \end{pmatrix} \right\} \tag{4.45}
\end{aligned}$$

Hence, the normal form of the system (4.41) is

$$\begin{aligned}
v_1' &= v_2 + \mathcal{O}(3), \\
v_2' &= \mathcal{O}(3), \\
v_3' &= \frac{1}{2}(2 - \lambda_2)\mu_2 v_1 v_3 + \mathcal{O}(3) \tag{4.46}
\end{aligned}$$

The solution of the truncated normal form passing through  $(v_1, v_2, v_3) = (v_{10}, v_{20}, v_{30})$  is

$$\begin{aligned}
v_1 &= v_{10} + v_{20}\tau, \quad v_2 = v_{20}, \\
v_3 &= v_{30} \exp \left[ \frac{1}{4}(2 - \lambda_2)\mu_2 \tau (2v_{10} + v_{20}\tau) \right]. \tag{4.47}
\end{aligned}$$

Since the phase space remains compact under the quadratic transformation that reduces the original system (4.41) to its normal form, (4.46), it follows that

1. For  $v_{10} \neq 0, v_{20} = v_{30} = 0$  the system tends asymptotically to the point  $(v_{10}, 0, 0)$ ; thus, such solutions remains very close to the critical point  $Q_5$ .
2. For  $v_{20} = 0$ , and  $v_{10}(2 - \lambda_2)\mu_2 > 0$ , the solution tends asymptotically to the point  $(v_{10}, 0, 0)$ ; thus, such solutions remains very close to the critical point  $Q_5$ .
3. If none of the above conditions holds and if we assume that the solutions are defined for all  $\tau$ -values, then, either  $|v_1| \rightarrow \infty$  or  $|v_3| \rightarrow \infty$  asymptotically (i.e., for  $\tau \rightarrow \pm\infty$ ) in contradiction with the compactness of the phase space. Thus, in this case, solutions should admit a finite time interval of existence.

In summary, the dynamical character  $Q_5$  is very sensible to the changes on the initial conditions. Thus, we argue that  $Q_5$  is unstable.

The basic observables (4.6), (4.7), (4.8) and (4.9) are given in terms of the Poincaré variables by

$$\begin{aligned}
\Omega_\phi &\equiv \frac{(x_r - y_r)(x_r + y_r)}{x_r^2 + y_r^2 + z_r^2 - 1} = -\frac{2\rho^2 \cos(2\theta) \sin^2(\psi)}{\cos(2\psi)\rho^2 - \rho^2 + 2}, \\
w_\phi &\equiv \frac{x_r^2 + y_r^2}{(x_r - y_r)(x_r + y_r)} = \sec(2\theta), \\
w_{\text{tot}} &\equiv \frac{x_r^2 + y_r^2}{x_r^2 + y_r^2 + z_r^2 - 1} = \frac{\rho^2(\cos(2\psi) - 1)}{\cos(2\psi)\rho^2 - \rho^2 + 2}, \\
q &\equiv \frac{4x_r^2 + 4y_r^2 + z_r^2 - 1}{2(x_r^2 + y_r^2 + z_r^2 - 1)} = \frac{2\cos(2\psi)\rho^2 - 2\rho^2 + 1}{\cos(2\psi)\rho^2 - \rho^2 + 2}.
\end{aligned} \tag{4.48}$$

In table 4.6 are displayed the values of the basic observables (4.6), (4.7), (4.8) and (4.9) for the singular points of the system (4.19) (case 2) as well as the solution types.

In order to illustrate the above analytical results we perform several numerical integrations.

In the figure 4.6 it is showed the Poincaré (global) phase-space of Model 2, for the parameter values  $\lambda_2 = -0.5$  and  $\mu_2 = 0.5$ . The critical point E (representing de Sitter solutions) is a local attractor for the points at the finite region. The singular points at infinity  $Q_{6,7,8,9}$  are local sources. In the figure are two lines (contained in the line  $y_r = z_r = 0$ ) that connects  $Q_6$  with  $D$  and  $Q_7$  with  $D$ , respectively. The points  $Q_{10,11,12}$  are saddles where as  $Q_{13}$  is a local sink.

In the figure 4.7 it is showed the Poincaré (global) phase-space of Model 2, for the parameter values  $\lambda_2 = 2.01 + \sqrt{30} > 2 + \sqrt{30}$  and  $\mu_2 = 0.5$ . The critical point E (representing de Sitter solutions) is, by theorem 16, a saddle for the points at the finite region. The singular points at infinity  $Q_{6,7,8,9,10}$  are local sources;  $Q_{12,13}$  are saddles and  $Q_{11}$  is a sink.

Table 4.6: Basic observables for the singular points of the system (4.19) (case 2). Solution types

Cr. P	$\Omega_\phi$	$w_\phi$	$w_{\text{tot}}$	$q$	Solution type
$Q_5$	0	1	0	$\frac{1}{2}$	dust-like
$Q_6$	$-\infty$	1	$-\infty$	$-\infty$	(unphys.) big-rip
$Q_7$	$-\infty$	1	$-\infty$	$-\infty$	(unphys.) big-rip
$Q_8$	0	$-\infty$	$-\infty$	$-\infty$	big-rip
$Q_9$	0	$+\infty$	$-\infty$	$-\infty$	big-rip
$Q_{10}$	0	$-\infty$	$-\frac{1}{18}(\lambda_2 - 2)^2$	$\frac{1}{12}(-\lambda_2^2 + 4\lambda_2 + 2)$	Accelerated for $ \lambda_2 - 2  > \sqrt{6}$ phantom for $ \lambda_2 - 2  > 3\sqrt{2}$
$Q_{11}$	0	$+\infty$	$-\frac{1}{18}(\lambda_2 - 2)^2$	$\frac{1}{12}(-\lambda_2^2 + 4\lambda_2 + 2)$	Accelerated for $ \lambda_2 - 2  > \sqrt{6}$ phantom for $ \lambda_2 - 2  > 3\sqrt{2}$
$Q_{12}$	$-\frac{1}{9}(\mu_2 - 2)^2$	1	$-\frac{1}{9}(\mu_2 - 2)^2$	$\frac{1}{6}(-\mu_2^2 + 4\mu_2 - 1)$	unphysical
$Q_{13}$	$-\frac{1}{9}(\mu_2 - 2)^2$	1	$-\frac{1}{9}(\mu_2 - 2)^2$	$\frac{1}{6}(-\mu_2^2 + 4\mu_2 - 1)$	unphysical

#### 4.2.2.4 Cosmological implications and discussion: Model 2

In this case, both real and physically meaningful critical points, namely D and E, have a stable manifold of smaller dimensionality than that of the phase-space. As was mentioned in subsection 4.2.2 the singular point D has a very small probability to be the late-time attractor of the system. By using the center manifold theory we have proved the stability of the de Sitter solution E for Power-law potential with  $\lambda_2 \leq 0$  and power-law dependent dark-matter particle mass. For  $\lambda_2 > 0$ ; E is locally unstable (saddle type). However, even if the cosmological evolution is managed to be attracted by these solutions, the coincidence problem will not be solved, since D represents a flat, non-accelerating universe dominated by dark matter, and E correspond to de Sitter universe completely dominated by dark energy. These critical points are located in the region where the scalar field and the Hubble parameter diverge. Divergencies in a cosmological scenario are represented as asymptotic states, in particular associated with the past and future asymptotic dynamics [136, 241, 242, 403, 404, 405, 406, 423]. In the present Model 2, due to the non-compactness of the phase-space, such a behavior can lead either to an asymptotic state acquired at infinite time, or to a singularity reached at a finite time. If  $H \rightarrow \infty$  or  $\rho_\phi \rightarrow \infty$  at  $t \rightarrow \infty$  then we acquire an eternally expanding universe, while if  $H \rightarrow \infty$  at  $t \rightarrow t_{BR} < \infty$  then the universe results to a Big Rip [434]. In order to give a complete picture of the physical model under consideration we have investigate the global phase space through Poincaré projection. We have obtained that the physical solutions at in-

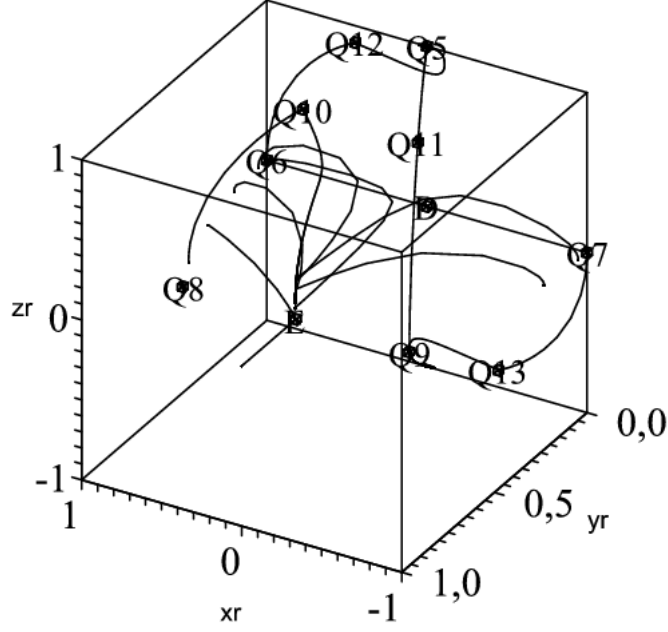


Figure 4.6: Poincaré (global) phase-space of Model 2, for the parameter values  $\lambda_2 = -0.5$  and  $\mu_2 = 0.5$ . The critical point E (representing de Sitter solutions) is a local attractor for the points at the finite region. The points at infinity  $Q_{6,7,8,9}$  are local sources;  $Q_{10,11,12}$  are saddles; and  $Q_{13}$  is a local sink.

finity are  $Q_{5,8,9,10,11}$ .  $Q_5$  represents a matter-dominated solution with effective equation of state  $w_{\text{tot}} = 0$  (dust-like) which is unstable by our previous analysis.  $Q_8$  and  $Q_9$  corresponds to initial big-rip singularities due that  $q \rightarrow -\infty$  and  $w_{\text{tot}} \rightarrow -\infty$ . That is, the points at infinity represents supper-accelerating ( $q \ll 0$ ) phantom solutions ( $w_{\text{tot}} \ll -1$ ). The solutions  $Q_{10,11}$  are accelerated for  $|\lambda_2 - 2| > \sqrt{6}$  and phantom for  $|\lambda_2 - 2| > 3\sqrt{2}$ . The unphysical solutions are  $Q_6$  and  $Q_7$  that represents unphysical big-rip singularities since  $q \rightarrow -\infty$  and  $w_{\text{tot}} \rightarrow -\infty$ , and  $Q_{12,13}$ . The last two singular points satisfy  $\Omega_\phi < 0$  for  $\mu_2 \neq 2$ . In the case  $\mu_2 = 2$  they reduce to  $Q_5$ . None of these solutions allows to solve the coincidence problem.

Therefore, power-law potentials with power-law-dependent dark-matter particle masses, cannot solve the coincidence problem.

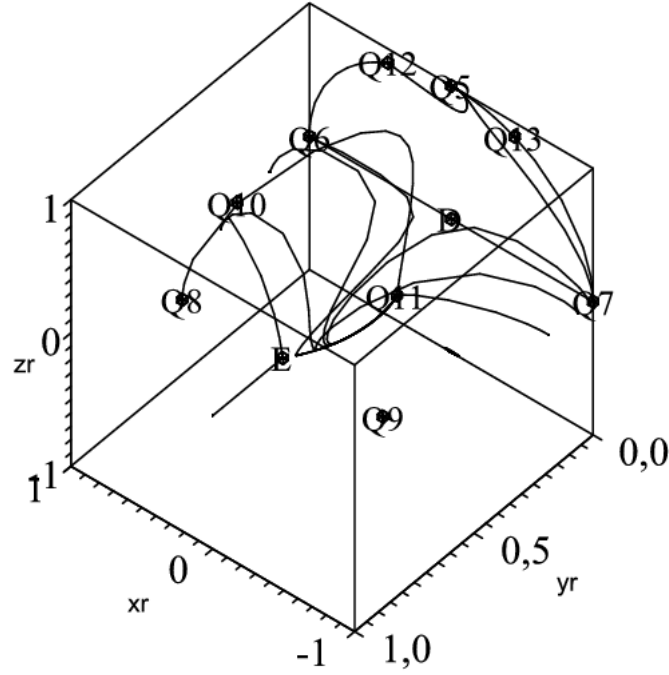


Figure 4.7: Poincaré (global) phase-space of Model 2, for the parameter values  $\lambda_2 = 2.01 + \sqrt{30}$  and  $\mu_2 = 0.5$ . The critical point E (representing de Sitter solutions) is a saddle point. The points at infinity  $Q_{6,7,8,9,10}$  are local sources;  $Q_{12,13}$  are saddles and  $Q_{11}$  is a sink.

### 4.2.3 Model 3: Power-law potential and exponentially-dependent dark-matter particle mass

In this case the autonomous system reads:

$$\begin{aligned} x' &= -3x + \frac{3}{2}x(1 - x^2 - y^2) - \frac{\lambda_2 y^2 z}{2} - \sqrt{\frac{3}{2}}\mu_1(1 + x^2 - y^2) \\ y' &= \frac{3}{2}y(1 - x^2 - y^2) - \frac{\lambda_2 x y z}{2}, \\ z' &= -x z^2. \end{aligned} \tag{4.49}$$

#### 4.2.3.1 Finite analysis

The real and physically meaningful critical points are

$$\begin{aligned} &(x_{c6} = 0, y_{c6} = 1, z_{c6} = 0), \\ &\left( x_{c7} = -\frac{\sqrt{\frac{3}{2}}}{\mu_1}, y_{c7} = \sqrt{1 - \frac{3}{2\mu_1^2}}, z_{c7} = 0 \right), \end{aligned} \tag{4.50}$$

and the necessary conditions for their existence are shown in table 4.7.

Table 4.7: The real and physically meaningful critical points of Model 3 and their behavior.

Cr. P.	$x_c$	$y_c$	$z_c$	Existence	Stable manifold	$\Omega_\sigma$	$w_{tot}$	Acc.
F	$x_{c6}$	$y_{c6}$	$z_{c6}$	Always	2-D	1	-1	Always
G	$x_{c7}$	$y_{c7}$	$z_{c7}$	$ \mu_1  > \sqrt{3}$	1-D	$1 - \frac{3}{\mu_1^2}$	-1	Always

In the model at hand, all critical points are non-hyperbolic and the dimensionality of their stable manifold is presented in table 4.7.

Additionally, we mention that there exists also an unphysical critical point H, with coordinates  $(x_{c8} = -\sqrt{\frac{2}{3}}\mu_1, y_{c8} = 0, z_{c8} = 0)$ . Its stable manifold is 2D if  $|\mu_1| > \sqrt{\frac{3}{2}}$ , and 1D if  $|\mu_1| < \sqrt{\frac{3}{2}}$ .

For the choice  $|\mu_1| > \sqrt{3}$ , the orbits initially below the stable manifold of G converge towards H. The orbits initially above this curve converge towards F. This behavior is depicted in fig. 4.8, which has arisen from numerical evolution using  $\lambda_2 = 1$  and  $\mu_1 = 1.8$ .

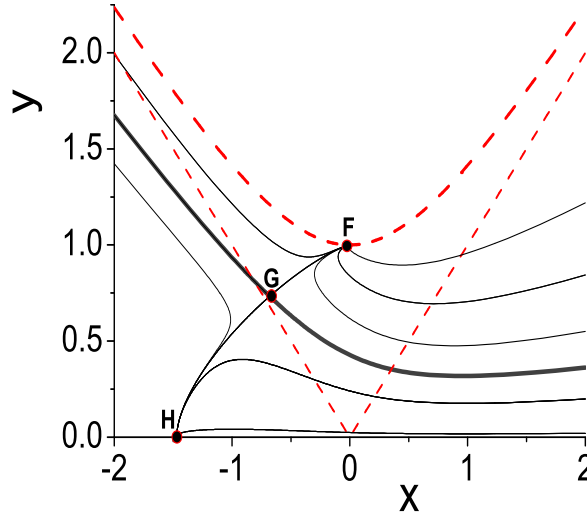


Figure 4.8:  $xy$ -projection of the phase-space of Model 3 for the parameter values  $\lambda_2 = 1$  and  $\mu_1 = 1.8$ . The stable manifold of G (thick curve) divides the physical part of the phase space (region bounded by the dashed (red) curves) in two regions. The orbits initially below this curve converge towards H, while those initially above this curve converge towards F. [Taken from [152]; published with permission of Elsevier B.V.].

If we restrict ourselves in the region  $|\mu_1| < \sqrt{\frac{3}{2}}$ , then the critical point G does not exist and thus there are not scaling solutions. In this case F is indeed the attractor for a positive-measure set of initial conditions. Moreover, there exist exceptional orbits contained on a 1D center manifold of F whose dynamical behavior cannot be anticipated



from the linear analysis. However, this scenario does not lead to a solution of the coincidence problem ( $\Omega_\phi = 1$  always).

#### 4.2.3.2 Stability of de Sitter solution for Power-law potential and exponentially-dependent dark-matter particle mass.

The singular point  $F$  represents the de Sitter solution for Power-law potential and power-law-dependent dark-matter particle mass. In this section we will analyze the stability of the center manifold of  $F$  for the vector field (4.49).

**Proposition 17** *For  $\lambda_2 < 0$ , the singular point  $F : (x_{c6} = 0, y_{c6} = 1, z_{c6} = 0)$  of the system (4.49) is locally asymptotically stable. For  $\lambda_2 > 0$ ,  $F$  is locally unstable (saddle type). For  $\lambda_2 = 0$ ,  $F$  is stable but not asymptotically stable.*

In order to translate  $F$  to the origin and transforming the linear part of the resulting vector field into its Jordan canonical form, we define new variables  $(u, v_1, v_2) \equiv \mathbf{x}$ , by the equations

$$u = z, v_1 = \frac{x}{\sqrt{6}\mu_1} + \frac{z\lambda_2}{6\sqrt{6}\mu_1}, v_2 = y - 1$$

so that

$$\begin{pmatrix} u' \\ v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} f(u, v_1, v_2) \\ g_1(u, v_1, v_2) \\ g_2(u, v_1, v_2) \end{pmatrix} \quad (4.51)$$

where

$$\begin{aligned} f(u, v_1, v_2) &= \frac{1}{6}u^2(u\lambda_2 - 6\sqrt{6}v_1\mu_1), \quad g_1(u, v_1, v_2) = \frac{\left(\frac{\lambda_2^3}{144\sqrt{6}} + \frac{\lambda_2^2}{36\sqrt{6}}\right)u^3}{\mu_1} + \\ &\left(\frac{1}{72}(-9v_1 - 1)\lambda_2^2 - \frac{v_1\lambda_2}{6}\right)u^2 + \left(\frac{v_1(9v_1+2)\lambda_2\mu_1}{2\sqrt{6}} - \frac{v_2(v_2+2)\lambda_2}{4\sqrt{6}\mu_1}\right)u - 3v_1^2(3v_1+1)\mu_1^2 + \frac{1}{2}v_2(v_2 - \\ &3v_1(v_2+2)), \text{ and } g_2(u, v_1, v_2) = \frac{1}{24}(v_2+1)(u^2\lambda_2^2 - 216v_1^2\mu_1^2) - \frac{3}{2}v_2^2(v_2+3). \end{aligned}$$

The system (4.51) is written in diagonal form

$$\begin{aligned} u' &= Cu + f(u, \mathbf{v}) \\ \mathbf{v}' &= P\mathbf{v} + \mathbf{g}(u, \mathbf{v}), \end{aligned} \quad (4.52)$$

where  $(u, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^2$ ,  $C$  is the zero  $1 \times 1$  matrix,  $P$  is a  $2 \times 2$  matrix with negative eigenvalues and  $f, \mathbf{g}$  vanish at  $\mathbf{0}$  and have vanishing derivatives at  $\mathbf{0}$ . The center manifold theorem 13 asserts that there exists a 1-dimensional invariant local center manifold  $W^c(\mathbf{0})$  of (4.52) tangent to the center subspace (the  $\mathbf{v} = \mathbf{0}$  space) at  $\mathbf{0}$ . Moreover,  $W^c(\mathbf{0})$  can be represented as

$$W^c(\mathbf{0}) = \{(u, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^2 : \mathbf{v} = \mathbf{h}(u), |u| < \delta\}; \mathbf{h}(\mathbf{0}) = \mathbf{0}, D\mathbf{h}(\mathbf{0}) = \mathbf{0}$$

for  $\delta$  sufficiently small (see definition 13). The restriction of (4.52) to the center manifold is (see definition 2.36)

$$u' = f(u, \mathbf{h}(u)). \quad (4.53)$$

According to Theorem 14, if the origin  $u = 0$  of (4.53) is stable (asymptotically stable) (unstable) then the origin of (4.52) is also stable (asymptotically stable) (unstable). Therefore, we have to find the local center manifold, i.e., the problem reduces to the computation of  $\mathbf{h}(u)$ .

Substituting  $\mathbf{v} = \mathbf{h}(u)$  in the second component of (4.52) and using the chain rule,  $\mathbf{v}' = D\mathbf{h}(u)u'$ , one can show that the function  $\mathbf{h}(u)$  that defines the local center manifold satisfies

$$D\mathbf{h}(u)[f(u, \mathbf{h}(u))] - P\mathbf{h}(u) - \mathbf{g}(u, \mathbf{h}(u)) = 0. \quad (4.54)$$

According to Theorem 15, equation (4.54) can be solved approximately by using an approximation of  $\mathbf{h}(u)$  by a Taylor series at  $u = 0$ . Since  $\mathbf{h}(0) = \mathbf{0}$  and  $D\mathbf{h}(0) = \mathbf{0}$ , it is obvious that  $\mathbf{h}(u)$  commences with quadratic terms. We substitute

$$\mathbf{h}(u) =: \begin{bmatrix} h_1(u) \\ h_2(u) \end{bmatrix} = \begin{bmatrix} a_1 u^2 + a_2 u^3 + O(u^4) \\ b_1 u^2 + b_2 u^3 + O(u^4) \end{bmatrix}$$

into (4.54) and set the coefficients of like powers of  $u$  equal to zero to find the unknowns  $a_1, b_1, \dots$

We find that the non-zero coefficients are

$$a_3 = \frac{\lambda_2^2}{108\sqrt{6}\mu_1}, \quad b_2 = \frac{\lambda_2^2}{72},$$

Therefore, (4.53) yields

$$u' = \frac{\lambda_2 u^3}{6} - \frac{\lambda_2^2 u^5}{108} + O(u^6). \quad (4.55)$$

Neglecting the error terms, this is a gradient-like equation (i.e.,  $u' = -\nabla U(u)$ ) with potential  $U(u) = \frac{1}{648}u^4\lambda_2(u^2\lambda_2 - 27)$  for which the origin is a degenerate minimum provided  $\lambda_2 < 0$  and a degenerated maximum provided  $\lambda_2 > 0$ . Thus, for  $\lambda_2 < 0$ , the origin  $u = 0$  of (4.55) is locally asymptotically stable. Hence, the origin  $\mathbf{u} = \mathbf{0}$  of the full three-dimensional system is asymptotically stable. For  $\lambda_2 > 0$  the origin is locally unstable (saddle type). In the same way as we proceeded in the proof of 16 can be proved the stability (but not asymptotic stability) of  $F$ . ■

#### 4.2.3.3 Analysis at infinity

Performing the transformation (4.33), the system (4.49) becomes

$$\begin{aligned}\rho' = & -\frac{3}{2}\rho^3 \sin^4 \psi - \frac{1}{2} \cos \theta \left( 2 \cos \psi \left( \cos^2 \psi + \lambda_2 \sin^2 \theta \sin^2 \psi \right) \rho^3 + \right. \\ & \left. + \sqrt{6}\mu_1 (1 - \rho^2)^{3/2} \right) \sin \psi + \cos(2\theta) \left( \frac{3}{2}\rho (\rho^2 - 1) \sin^2 \psi + \right. \\ & \left. - \frac{1}{2}\mu_1 \rho^2 \sqrt{6 - 6\rho^2} \cos \theta \sin^3 \psi \right),\end{aligned}\quad (4.56)$$

$$\begin{aligned}\theta' = & 3 \cos \theta \sin \theta + \frac{\mu_1 \sqrt{6 - 6\rho^2} \csc \psi \sin \theta}{2\rho} + \\ & + \frac{\rho \cos(2\theta) \left( \lambda_2 \rho \cos \psi - \mu_1 \sqrt{6 - 6\rho^2} \right) \sin \psi \sin \theta}{2(\rho^2 - 1)},\end{aligned}\quad (4.57)$$

$$\begin{aligned}\psi' = & \frac{3\rho^2 \cos \psi \sin^3 \psi}{2(\rho^2 - 1)} \\ & + \frac{\cos \theta \cos \psi \left( (\lambda_2 - 2) \cos \psi \sin^2 \psi \rho^3 + \sqrt{6}\mu_1 (1 - \rho^2)^{3/2} \right)}{2\rho(\rho^2 - 1)} + \\ & + \cos(2\theta) \left( \frac{\rho \cos \theta \cos \psi \left( \mu_1 \sqrt{6 - 6\rho^2} - \lambda_2 \rho \cos \psi \right) \sin^2 \psi}{2(\rho^2 - 1)} + \right. \\ & \left. - \frac{3}{2} \cos \psi \sin(\psi) \right).\end{aligned}\quad (4.58)$$

In the limit  $\rho \rightarrow 1$ , the leading terms in (4.56)-(4.58) are

$$\rho' \rightarrow -\frac{1}{2} \sin \psi \left( 3 \sin^3 \psi + 2 \cos \theta \cos \psi \left( \cos^2 \psi + \lambda_2 \sin^2 \theta \sin^2 \psi \right) \right), \quad (4.59)$$

$$\theta' \rightarrow -\frac{\lambda_2 \cos(2\theta) \cos \psi \sin \theta \sin \psi}{2(1 - \rho^2)}, \quad (4.60)$$

$$\psi' \rightarrow \frac{\cos \psi (\cos \theta (\cos(2\theta) \lambda_2 - \lambda_2 + 2) \cos \psi - 3 \sin \psi) \sin^2 \psi}{2(1 - \rho^2)}. \quad (4.61)$$

The radial equation does not contain the radial coordinate, so the singular points can be obtained using the angular equations only. Setting  $\theta' = 0, \psi' = 0$ , we obtain the singular points which are listed in table 4.8. The stability of these points is studied by analyzing first the stability of the angular coordinates and then deducing, from the sign of equation (4.59), the stability on the radial direction.

Table 4.8: Asymptotic singular points of the system (4.49) (case 3) and their stability. We use the notations  $\alpha = \frac{3\sqrt{2}}{\sqrt{22-4\lambda_2+\lambda_2^2}}$ ,  $\epsilon = \text{sign}(\lambda_2)$ ,  $\delta = \text{sign}(-26 + 4\lambda_2 + \lambda_2^2)$ . NH stands for nonhyperbolic.

Cr. P	Coordinates $\theta, \psi, x_r, y_r, z_r$	Eigenvalues	$\rho'$	Stability
$Q_{14}$	$0, 0, 0, 0, 1$	$0, 0$	$0$	NH; 3D center manifold
$Q_{15}$	$0, \frac{\pi}{2}, 1, 0, 0$	$0, +\infty$	$-\frac{3}{2}$	NH; 2D unstable manifold
$Q_{16}$	$\pi, \frac{\pi}{2}, -1, 0, 0$	$0, +\infty$	$-\frac{3}{2}$	NH; 2D unstable manifold
$Q_{17}$	$\frac{\pi}{4}, \frac{\pi}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0$	$0, +\infty$	$-\frac{3}{2}$	NH; 2D unstable manifold
$Q_{18}$	$\frac{3\pi}{4}, \frac{\pi}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0$	$0, +\infty$	$-\frac{3}{2}$	NH; 2D unstable manifold
$Q_{19}$	$0, \cos^{-1}\left(\frac{3}{\sqrt{13}}\right), \frac{2}{\sqrt{13}}, 0, \frac{3}{\sqrt{13}}$	$-\epsilon\infty, -\infty$	$< 0$	saddle
$Q_{20}$	$\frac{\pi}{4}, \cos^{-1}(\alpha), \frac{\alpha \lambda_2-2 }{6}, \frac{\alpha \lambda_2-2 }{6}, \alpha$	$+\infty, \delta\infty$ if $\lambda_2 > 2$ $\epsilon\infty, -\infty$ if $\lambda_2 < 2$	$< 0$	source if $\lambda_2 > 2 + \sqrt{30}$ saddle, otherwise
$Q_{21}$	$\frac{3\pi}{4}, \cos^{-1}(\alpha), -\frac{\alpha \lambda_2-2 }{6}, \frac{\alpha \lambda_2-2 }{6}, \alpha$	$-\infty, -\infty$ if $\lambda_2 > 2$ $-\epsilon\infty, \delta\infty$ if $\lambda_2 < 2$	$> 0$ if $\lambda_2 > -1$ and $\lambda_2 \neq 2$ , $< 0$ if $\lambda_2 < -1$	sink, if $\lambda_2 > -1$ , and $\lambda_2 \neq 2$ , saddle, otherwise

In table 4.9 are displayed the values of the basic observables (4.6), (4.7), (4.8) and (4.9) for the singular points of the system (4.49) (case 3) as well as the solution types.

Table 4.9: Basic observables for the singular points of the system (4.49) (case 3). Solution types

Cr. P	$\Omega_\phi$	$w_\phi$	$w_{\text{tot}}$	$q$	Solution type
$Q_{14}$	0	1	0	$\frac{1}{2}$	dust-like
$Q_{15}$	$-\infty$	1	$-\infty$	$-\infty$	(unphys.) big-rip
$Q_{16}$	$-\infty$	1	$-\infty$	$-\infty$	(unphys.) big-rip
$Q_{17}$	0	$-\infty$	$-\infty$	$-\infty$	big-rip
$Q_{18}$	0	$+\infty$	$-\infty$	$-\infty$	big-rip
$Q_{19}$	$-\frac{4}{9}$	1	$-\frac{4}{9}$	$-\frac{1}{6}$	unphysical
$Q_{20}$	0	$-\infty$	$-\frac{1}{18}(\lambda_2 - 2)^2$	$\frac{1}{12}(-\lambda_2^2 + 4\lambda_2 + 2)$	Accelerated for $ \lambda_2 - 2  > \sqrt{6}$ phantom for $ \lambda_2 - 2  > 3\sqrt{2}$
$Q_{21}$	0	$+\infty$	$-\frac{1}{18}(\lambda_2 - 2)^2$	$\frac{1}{12}(-\lambda_2^2 + 4\lambda_2 + 2)$	Accelerated for $ \lambda_2 - 2  > \sqrt{6}$ phantom for $ \lambda_2 - 2  > 3\sqrt{2}$

In order to perform the numerical experiments for the system (4.56)-(4.58) it is useful to rewrite the system in the cartesian coordinates  $x_r, y_r, z_r$ . The system (4.56)-(4.58) becomes

$$x'_r = \frac{1}{2} (2x_r^2 z_r^3 + 3x_r (-x_r^2 + y_r^2 + 1) z_r^2 + \lambda_2 (2x_r^2 - 1) y_r^2 z_r + \\ + 3x_r (x_r^2 + y_r^2 - 1) (2y_r^2 + 1)) - \sqrt{\frac{3}{2}} \mu_1 (x_r^2 - 1) w (2y_r^2 + z_r^2 - 1),$$

$$y'_r = \frac{1}{2} y_r ((6y_r^2 - 3(z_r^2 + 1)) x_r^2 + z_r (2\lambda_2 y_r^2 + 2z_r^2 - \lambda_2) x_r \\ + 3(y_r^2 - 1) (2y_r^2 + z_r^2 - 1)) - \sqrt{\frac{3}{2}} \mu_1 x_r y_r w (2y_r^2 + z_r^2 - 1),$$

$$\begin{aligned}
z'_r &= \frac{1}{2}z_r \left( (6y_r^2 - 3z_r^2 + 3) x_r^2 + 2z_r (\lambda_2 y_r^2 + z_r^2 - 1) x_r + \right. \\
&\quad \left. + 3y_r^2 (2y_r^2 + z_r^2 - 1) \right) - \sqrt{\frac{3}{2}} \mu_1 x_r z_r w (2y_r^2 + z_r^2 - 1) \\
w' &= \frac{1}{2}w \left( (6y_r^2 - 3z_r^2 + 3) x_r^2 + 2z_r (\lambda_2 y_r^2 + z_r^2) x_r + \right. \\
&\quad \left. + 3y_r^2 (2y_r^2 + z_r^2 - 1) \right) - \sqrt{\frac{3}{2}} w^2 \mu_1 x_r (2y_r^2 + z_r^2 - 1)
\end{aligned} \tag{4.62}$$

where we have used the time re-scaling

$$d\tau \rightarrow \frac{d\tau}{1 - \rho^2}$$

which leave invariant the orbits of the phase-space and the time direction (see theorem 4) and we have introduced the auxiliary variable  $w = \sqrt{1 - x_r^2 - y_r^2 - z_r^2}$  to avoid that the numerical procedure becomes complex-valued at the singular points.

To examine the stability of  $Q_{14}$  we perform the linear coordinate transformation  $u_1 = x_r$ ,  $u_2 = y_r$ ,  $u_3 = z_r - 1$ ,  $u_4 = w$  and Taylor expand the system (4.62) around the origin up to third order in the vector norm to obtain the approximated system

$$\begin{aligned}
u'_1 &= u_1^2 + 3u_3u_1 - \frac{u_2^2\lambda_2}{2} + \sqrt{6}u_3u_4\mu_1 + \mathcal{O}(3), \\
u'_2 &= -\frac{1}{2}u_2(6u_3 + u_1(\lambda_2 - 2)) + \mathcal{O}(3), \\
u'_3 &= 2u_1u_3 + \mathcal{O}(3), \\
u'_4 &= u_1u_4 + \mathcal{O}(3),
\end{aligned} \tag{4.63}$$

where  $\mathcal{O}(3)$  denotes  $\mathcal{O}(\|(u_1, u_2, u_3, u_4)\|^3)$  defined in a neighborhood of the origin contained in the region

$$\left\{ 0 \leq u_1^2 + u_2^2 \leq 1, -1 - \sqrt{1 - u_1^2 - u_2^2} \leq u_3 \leq -1 + \sqrt{1 - u_1^2 - u_2^2} \leq 0, u_4 > 0 \right\}.$$

Observe that the variables  $u_i, i = 1 \dots 4$  are not independent since  $u_1^2 + u_2^2 + u_3(u_3 + 2) + u_4^2 = 0$ . Thus, one is able to eliminate one variable. From (4.63)(c) and (4.63)(d) follows that  $u_3 \propto u_4^2$ . By substituting back this relation on (4.63) and neglecting the terms  $\mathcal{O}(\|(u_1, u_2, u_4)\|^3)$  we obtain the reduce (decoupled) 3-dimensional system

$$\begin{aligned}
u'_1 &= u_1^2 - \frac{u_2^2\lambda_2}{2}, \\
u'_2 &= -\frac{1}{2}(\lambda_2 - 2)u_1u_2, \\
u'_4 &= u_1u_4,
\end{aligned} \tag{4.64}$$

that represents very accurately the nonlinear dynamics.<sup>2</sup>

The system (4.64) admits the implicit solution

$$\begin{aligned} u_1 &= \pm \sqrt{c_1 u_2^{-\frac{4}{\lambda_2-2}} + u_2^2}, \\ \tau &= c_2 \mp \frac{u_2^{\frac{2}{\lambda_2-2}} {}_2F_1\left(\frac{1}{\lambda_2}, \frac{1}{2}; 1 + \frac{1}{\lambda_2}; -\frac{u_2^{\frac{2\lambda_2}{\lambda_2-2}}}{c_1}\right)}{\sqrt{c_1}}, \\ u_4 &= u_2^{-\frac{2}{\lambda_2-2}} c_3 \end{aligned} \quad (4.65)$$

By examining the asymptotic behavior of solutions (4.65) we obtain that for  $\lambda_2 > 2$ , the variable  $u_2$  cannot approach asymptotically to zero since otherwise this would imply the divergence of  $u_1$  and  $u_4$  in contradiction with the compactness of the phase space. For  $0 < \lambda_2 < 2$ , the origin is approached as  $\tau \rightarrow -\infty$  in the case of  $u_1(0) > 0$  and it is approached as  $\tau \rightarrow +\infty$  in the case of  $u_1(0) < 0$ . For  $\lambda_2 < 0$ ,

$$\frac{u_2^{\frac{2}{\lambda_2-2}} {}_2F_1\left(\frac{1}{\lambda_2}, \frac{1}{2}; 1 + \frac{1}{\lambda_2}; -\frac{u_2^{\frac{2\lambda_2}{\lambda_2-2}}}{c_1}\right)}{\sqrt{c_1}} \approx \frac{u_2^{\frac{2}{\lambda_2-2}}}{\sqrt{c_1}} - \frac{u_2^{2+\frac{6}{\lambda_2-2}}}{2(\lambda_2+1)c_1^{3/2}} + \text{h.o.t.} \rightarrow +\infty$$

as  $u_2 \rightarrow 0$ . This result can be interpreted as follows. For  $\lambda_2 < 0$  and  $u_1(0) > 0$  the origin is approached as  $\tau \rightarrow -\infty$  and for  $\lambda_2 < 0$  and  $u_1(0) < 0$  the origin is approached as  $\tau \rightarrow +\infty$ , which means that  $Q_{14}$  is saddle like. In this way we have proved the instability of  $Q_{14}$ .

We obtain by an explicit calculation that the center manifold of  $Q_{17,18}$  is given up to fourth order by the graph  $x_r \pm y_r \mp \sqrt{2} = \mp \sqrt{2}u^2$ ,  $z_r = 0$ ,  $w = 0$  where  $u \equiv y_r - \frac{\sqrt{2}}{2}$ . The dynamics on the center manifold is given by

$$u' = -6u^3 + \mathcal{O}(5).$$

From this follows that the center manifold is stable since the origin is a degenerate minimum of the potential  $U(u) = \frac{3u^4}{2}$ . Hence,  $Q_{17,18}$  are of saddle type.

By an explicit calculation we find that the center manifolds of  $Q_{15,16}$  are  $x_r = \pm \left(1 - \frac{y_r^2}{2}\right) + \mathcal{O}(y_r)^3 = \pm \sqrt{1+y^2}$ ,  $z_r = w = 0$ . The dynamics on the center manifolds of  $Q_{15,16}$  are given by  $u' = \frac{3}{2}u^2 + \mathcal{O}(u)^5$ , where  $u = y_r$ . From this follows that  $Q_{15,16}$  are local sources since their center manifolds are unstable (the origin is a degenerate local maximum of the potential function  $U(u) = -\frac{3u^4}{8}$ ).

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<sup>2</sup>Another argument in favor to neglect the contribution of  $u_3$  to the nonlinear dynamics is that  $u_1^2 + u_2^2 + u_3(u_3 + 2) + u_4^2 = 0$  is an exact formula. Thus, by taking the total derivative of  $u_1^2 + u_2^2 + u_3(u_3 + 2) + u_4^2$  we obtain  $\frac{d}{d\tau} [u_1^2 + u_2^2 + u_3(u_3 + 2) + u_4^2] = 4u_1u_3 + \mathcal{O}(\|(u_1, u_2, u_3, u_4)\|^3)$ , which means that  $u_1u_3 = \mathcal{O}(\|(u_1, u_2, u_3, u_4)\|^3)$ , and then  $u_3' = \mathcal{O}(\|(u_1, u_2, u_3, u_4)\|^3)$ . This result is consistent with  $u_3 \propto u_4^2$ .

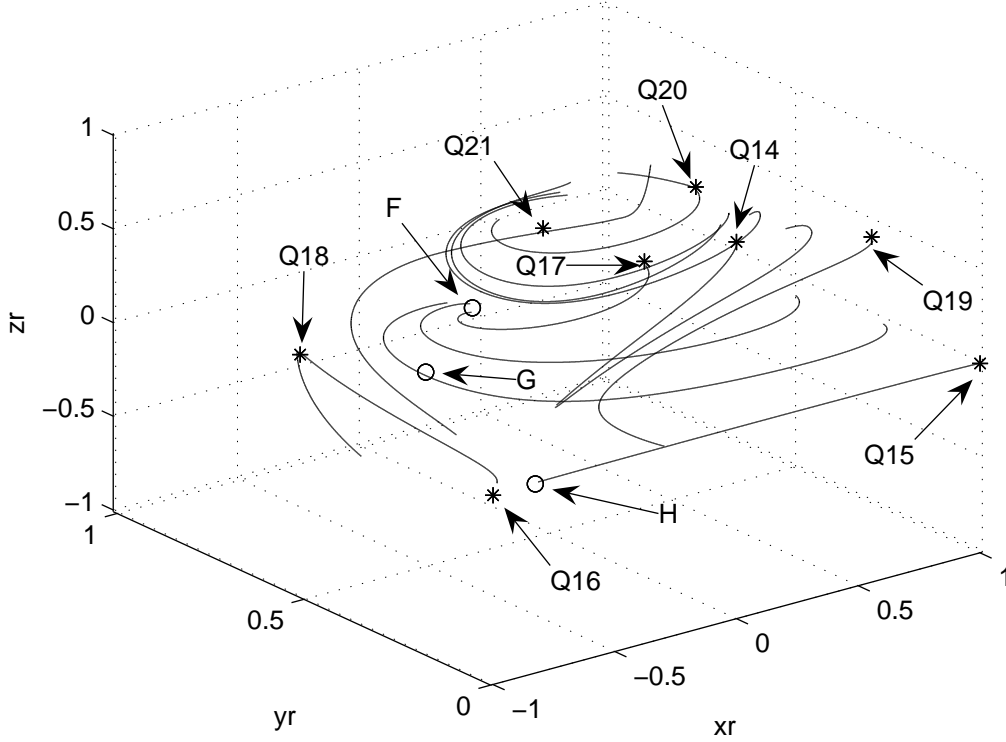


Figure 4.9: Poincaré (global) phase-space of Model 3, for the parameter values  $\lambda_2 = -0.01$  and  $\mu_1 = 1.8$ . For this choice of parameters  $Q_{15}$  and  $Q_{16}$  are local sources;  $Q_{14}$  is unstable (of saddle type);  $Q_{17,18,19,20}$  are saddles in the infinite region;  $G, H$  are saddles in the finite region;  $Q_{21}$  is a sink in the infinite region and  $F$  is locally asymptotically stable.

In the figure 4.9 are displayed several trajectories in the Poincaré (global) phase-space of Model 3, for the parameter values  $\lambda_2 = -0.01$  and  $\mu_1 = 1.8$ . For this choice of parameters  $Q_{15}$  and  $Q_{16}$  are local sources;  $Q_{14}$  is unstable (of saddle type);  $Q_{17,18,19,20}$  are saddles in the infinite region;  $G, H$  are saddles in the finite region;  $Q_{21}$  is a sink in the infinite region and  $F$  is locally asymptotically stable. There is one orbit joining  $Q_{15}$  and  $H$ ; and one orbit joining  $Q_{15}$  with  $Q_{19}$ .

#### 4.2.3.4 Cosmological implications and discussion: Model 3

In this model we see that the critical point  $F$  exists always, while  $G$  exists only for  $|\mu_1| > \sqrt{3}$ . However, in both cases the stable manifold is of smaller dimensionality than that of the phase-space. However, by an explicit computation of the center manifold we have proved that for  $\lambda_2 \leq 0$  the singular point  $F$  (corresponding to a de Sitter solution) is stable. Furthermore, in order to avoid the treatment of unphysical attracting states we have to impose the additional constraint  $|\mu_1| < \sqrt{\frac{3}{2}}$ . For this choice of parameters,  $G$  does not exist and thus there are not scaling solutions, while  $F$  is the attractor for a positive-



measure set of initial conditions. Point F corresponds to a dark-energy dominated de Sitter universe, while G to a flat accelerating universe with  $\Omega_\phi = 1 - \frac{3}{\mu_1^2}$ , that is with  $0 < \Omega_\phi < 1$  in the region that it exists. In both points the phantom field diverges. However, even if G possesses  $0 < \Omega_\phi < 1$ , it can not solve the coincidence problem since it is not a relevant late-time attractor.

In order to give a complete picture of the physical model under consideration we have investigate the global phase space through Poincaré projection. We have obtained that the physical solutions at infinity are  $Q_{14,17,18,20,21}$ .  $Q_{14}$  represents a matter-dominated solution with effective equation of state  $w_{\text{tot}} = 0$  (dust-like) which is unstable by our previous analysis.  $Q_{17}$  and  $Q_{18}$  corresponds to initial big-rip singularities due that  $q \rightarrow -\infty$  and  $w_{\text{tot}} \rightarrow -\infty$ . That is, the points at infinity represents super-accelerating ( $q \ll 0$ ) phantom solutions ( $w_{\text{tot}} \ll -1$ ). The solutions  $Q_{20,21}$  are accelerated for  $|\lambda_2 - 2| > \sqrt{6}$  and phantom for  $|\lambda_2 - 2| > 3\sqrt{2}$ . The unphysical solutions are  $Q_{15}$  and  $Q_{16}$  that represents unphysical big-rip singularities since  $q \rightarrow -\infty$  and  $w_{\text{tot}} \rightarrow -\infty$ , and  $Q_{12,13}$ . The singular point  $Q_{19}$  satisfy  $\Omega_\phi < 0$ ; thus, it is unphysical. None of these solutions allows to solve the coincidence problem.

In summary, power-law potentials with exponentially-dependent dark-matter particle masses cannot solve or even alleviate the coincidence problem.

#### 4.2.4 Model 4: Exponential potential and power-law-dependent dark-matter particle mass

In this case the autonomous system writes:

$$\begin{aligned} x' &= -3x + \frac{3}{2}x(1 - x^2 - y^2) - \sqrt{\frac{3}{2}}\lambda_1 y^2 - \frac{\mu_2}{2}z(1 + x^2 - y^2) \\ y' &= \frac{3}{2}y(1 - x^2 - y^2) - \sqrt{\frac{3}{2}}\lambda_1 xy \\ z' &= -xz^2. \end{aligned} \tag{4.66}$$

$$\tag{4.67}$$

##### 4.2.4.1 Finite analysis

The real and physically meaningful critical points are

$$\begin{aligned} &(x_{c9} = 0, y_{c9} = 0, z_{c9} = 0), \\ &\left( x_{c10} = -\frac{\lambda_1}{\sqrt{6}}, y_{c10} = \sqrt{1 + \frac{\lambda_1^2}{6}}, z_{c10} = 0 \right), \end{aligned} \tag{4.68}$$

and in table 4.10 we present the necessary conditions for their existence. The aforementioned critical points are non-hyperbolic since at least one eigenvalue of  $\mathbf{Q}$  is always zero. Linear analysis is not conclusive in these cases, but information about the dimensionality

Table 4.10: The real and physically meaningful critical points of Model 4 and their behavior.

Cr. P.	$x_c$	$y_c$	$z_c$	Existence	Stable manifold	$\Omega_\sigma$	$w_{tot}$	Acc.
I	$x_{c9}$	$y_{c9}$	$z_{c9}$	Always	1-D	0	0	Never
J	$x_{c10}$	$y_{c10}$	$z_{c10}$	Always	2-D	1	$-\frac{1}{3}(3 + \lambda_1^2)$	Always

of the stable manifold can be obtained by applying the center manifold theorem [359]. The corresponding results are shown in table 4.10. Both I and J cannot solve the coincidence problem ( $\Omega_\phi = 1$ ). In the next section we examine the stability of the phantom solution for this case.

In order to acquire a more transparent picture of the phase-space behavior, we evolve the system numerically for  $\lambda_1 = 1$  and  $\mu_2 = 1.8$  and we depict the results in fig. 4.10.

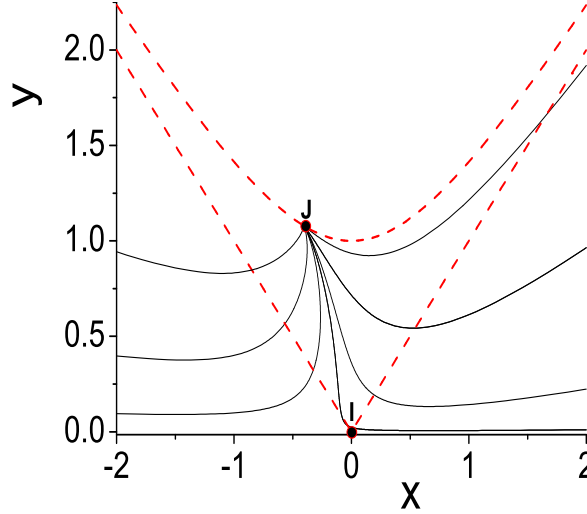


Figure 4.10:  $xy$ -projection of the phase-space of Model 4 for the parameter values  $\lambda_1 = 1$  and  $\mu_2 = 1.8$ . The critical point J (corresponding to a super-accelerating universe) attracts all the orbits in this invariant set. The dashed (red) curves bound the physical part of the phase space, that is corresponding to  $0 \leq \Omega_\phi \leq 1$ . [Taken from [152]; published with permission of Elsevier B.V.].

#### 4.2.4.2 Stability analysis of the phantom dominated solution for the exponential potential and power-law-dependent dark-matter particle mass

In this section we examine the stability of the phantom dominated solution for the exponential potential and power-law-dependent dark-matter particle mass through the stability analysis of its center manifold.

**Proposition 18** *The singular point  $J : \left( x_{c10} = -\frac{\lambda_1}{\sqrt{6}}, y_{c10} = \sqrt{1 + \frac{\lambda_1^2}{6}}, z_{c10} = 0 \right)$  of the system (4.67) is stable but is not asymptotically stable.*

**Proof.**

In order to translate  $J$  to the origin and transforming the linear part of the resulting vector field into its Jordan canonical form, we define new variables  $(u, v_1, v_2) \equiv \mathbf{x}$ , by the equations

$$u = z, v_1 = y - \frac{\sqrt{\lambda_1^2 + 6}}{\sqrt{6}}, v_2 = \frac{\sqrt{\lambda_1^2 + 6}y + x\lambda_1 - \sqrt{6}}{\lambda_1}$$

so that

$$\begin{pmatrix} u' \\ v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}(-\lambda_1^2 - 6) & 0 \\ 0 & 0 & -\lambda_1^2 - 3 \end{pmatrix} \begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} f(u, v_1, v_2) \\ g_1(u, v_1, v_2) \\ g_2(u, v_1, v_2) \end{pmatrix} \quad (4.69)$$

where

$$\begin{aligned} f(u, v_1, v_2) &= u^2 \left( \frac{\sqrt{\lambda_1^2 + 6}v_1}{\lambda_1} - v_2 + \frac{\lambda_1}{\sqrt{6}} \right), g_1(u, v_1, v_2) = -\frac{9v_1^3}{\lambda_1^2} - 3v_1^3 + \frac{3v_2\sqrt{\lambda_1^2 + 6}v_1}{\lambda_1} - \\ &\frac{3\sqrt{\frac{3}{2}}\sqrt{\lambda_1^2 + 6}v_1^2}{\lambda_1^2} - \sqrt{6}\sqrt{\lambda_1^2 + 6}v_1^2 - \frac{3v_2^2v_1}{2} + \sqrt{\frac{3}{2}}v_2\lambda_1v_1 + \frac{3\sqrt{6}v_2v_1}{\lambda_1} - \frac{1}{2}\sqrt{\frac{3}{2}}v_2^2\sqrt{\lambda_1^2 + 6}, \text{ and} \\ g_2(u, v_1, v_2) &= -\frac{3v_2^3}{2} + \sqrt{\frac{3}{2}}\lambda_1v_2^2 - \frac{1}{2}u\mu_2v_2^2 + \frac{3v_1\sqrt{\lambda_1^2 + 6}v_2^2}{\lambda_1} - \frac{3\sqrt{\frac{3}{2}}v_2^2}{\lambda_1} - 3v_1^2v_2 + \frac{u\lambda_1\mu_2v_2}{\sqrt{6}} + \\ &\frac{uv_1\sqrt{\lambda_1^2 + 6}\mu_2v_2}{\lambda_1} - 3\sqrt{\frac{3}{2}}v_1\sqrt{\lambda_1^2 + 6}v_2 + \frac{3\sqrt{6}v_1\sqrt{\lambda_1^2 + 6}v_2}{\lambda_1^2} - \frac{9v_1^2v_2}{\lambda_1^2} - \frac{3uv_1^2\mu_2}{\lambda_1^2} - \frac{9\sqrt{6}v_1^2}{\lambda_1^3}. \end{aligned}$$

The system (4.69) is written in diagonal form

$$\begin{aligned} u' &= Cu + f(u, \mathbf{v}) \\ \mathbf{v}' &= P\mathbf{v} + \mathbf{g}(u, \mathbf{v}), \end{aligned} \quad (4.70)$$

where  $(u, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^2$ ,  $C$  is the zero  $1 \times 1$  matrix,  $P$  is a  $2 \times 2$  matrix with negative eigenvalues and  $f, \mathbf{g}$  vanish at  $\mathbf{0}$  and have vanishing derivatives at  $\mathbf{0}$ . The center manifold theorem 13 asserts that there exists a 1-dimensional invariant local center manifold  $W^c(\mathbf{0})$  of (4.70) tangent to the center subspace (the  $\mathbf{v} = \mathbf{0}$  space) at  $\mathbf{0}$ . Moreover,  $W^c(\mathbf{0})$  can be represented as

$$W^c(\mathbf{0}) = \{(u, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^2 : \mathbf{v} = \mathbf{h}(u), |u| < \delta\}; \mathbf{h}(0) = \mathbf{0}, D\mathbf{h}(0) = \mathbf{0}$$

for  $\delta$  sufficiently small (see definition 13).

The equations for the center manifold of the origin reduces to

$$\begin{aligned}
& \frac{36(\lambda_1^2 + 3)h_1^3}{\lambda_1} + \left( \frac{6\sqrt{6}\sqrt{\lambda_1^2 + 6}(2\lambda_1^2 + 3)}{\lambda_1} - 36\sqrt{\lambda_1^2 + 6}h_2 \right) h_1^2 + \\
& + \left( 18\lambda_1 h_2^2 - 6\sqrt{6}(\lambda_1^2 + 6)h_2 + 6\lambda_1(\lambda_1^2 + 6) \right) h_1 + \\
& + 3\sqrt{6}\lambda_1\sqrt{\lambda_1^2 + 6}h_2^2 + \\
& + \left( 2\sqrt{6}\lambda_1^2 u^2 + 12\sqrt{\lambda_1^2 + 6}h_1 u^2 - 12\lambda_1 h_2 u^2 \right) h_1' = 0, \\
& 9\lambda_1^2 h_2^3 + 3\lambda_1 \left( -\sqrt{6}\lambda_1^2 + u\mu_2\lambda_1 + 3\sqrt{6} \right) h_2^2 + \\
& + \lambda_1^2 \left( 6\lambda_1^2 - \sqrt{6}u\mu_2\lambda_1 + 18 \right) h_2 + \\
& + h_1^2 \left( 18u\mu_2 + 18(\lambda_1^2 + 3)h_2 + \frac{54\sqrt{6}}{\lambda_1} \right) + \\
& + h_1 \left( 3\sqrt{\lambda_1^2 + 6} \left( 3\sqrt{6}\lambda_1^2 - 2u\mu_2\lambda_1 - 6\sqrt{6} \right) h_2 - 18\lambda_1\sqrt{\lambda_1^2 + 6}h_2^2 \right) + \\
& + \left( \sqrt{6}u^2\lambda_1^3 - 6u^2h_2\lambda_1^2 + 6u^2\sqrt{\lambda_1^2 + 6}h_1\lambda_1 \right) h_2' = 0. \tag{4.71}
\end{aligned}$$

We obtain, using a Taylor series at  $u = 0$ , that the solution of (18) satisfying  $\mathbf{h}(0) = 0$ ,  $D\mathbf{h}(0) = \mathbf{0}$  is the trivial solution to arbitrary order. This means that the center manifold of  $J$  is a small segment contained in the  $z$ -axis.

In order to examine the stability of the origin for the flow of (4.71) we proceed as follows. Using spherical coordinates

$$u = r \cos \varphi \sin \theta, v_1 = r \sin \theta \sin \varphi, v_2 = r \cos \theta \tag{4.72}$$

and taking the limit  $r \rightarrow 0$  the angular equations  $\theta', \varphi'$  become

$$\begin{aligned}
\varphi' & \rightarrow -\frac{1}{4}(\lambda_1^2 + 6) \sin(2\varphi), \\
\theta' & \rightarrow \frac{1}{8} \left( 3(\lambda_1^2 + 2) + (\lambda_1^2 + 6) \cos(2\varphi) \right) \sin(2\theta). \tag{4.73}
\end{aligned}$$

Solving the approximate equations (4.73) we obtain

$$\begin{aligned}
\theta(\tau) & = \tan^{-1} \left( e^{(\lambda_1^2 + 3)\tau + 2c_2} \sqrt{1 + e^{(-\lambda_1^2 - 6)\tau + 4c_1}} \right), \\
\varphi(\tau) & = \tan^{-1} \left( e^{2c_1 - \frac{1}{2}(\lambda_1^2 + 6)\tau} \right), \tag{4.74}
\end{aligned}$$

where  $c_1$  and  $c_2$  are integration constants.

By Taylor expanding the radial equation around  $r = 0$  we obtain the equation

$$r' = -\frac{1}{2}r \left( 2(\lambda_1^2 + 3) \cos^2(\theta) + (\lambda_1^2 + 6) \sin^2(\theta) \sin^2(\varphi) \right) + O(r^2). \quad (4.75)$$

By substituting the first order solution (4.74) into the equation (4.75) and solving the resulting equation we obtain

$$r(\tau) = e^{(-\lambda_1^2 - 3)\tau} \sqrt{1 + e^{2(\lambda_1^2 + 3)\tau + 4c_2} + e^{\tau\lambda_1^2 + 4(c_1 + c_2)} c_3}, \quad (4.76)$$

where  $c_3$  is an integration constant. Substituting (4.74) and (4.76) in (4.72) and taking the limit as  $\tau \rightarrow +\infty$  we obtain  $u \rightarrow u_0 = e^{2c_2} c_3$ ,  $v_1 \rightarrow 0$ ,  $v_2 \rightarrow 0$  where  $u_0 = u(0)$ . Let be  $\epsilon > 0$  an arbitrary number. Then there exists a  $\delta > 0$ , such that  $\delta < \epsilon$ . Let us consider the solution with initial value  $u(0) = u_0$ ,  $v_1(0) = v_{10}$ ,  $v_2(0) = v_{20}$ , with  $u_0^2 + v_{10}^2 + v_{20}^2 < \delta^2$ . Since  $u \rightarrow u_0$ , satisfying  $|u_0| < \delta$ , then the solution,  $\mathbf{x}(\tau, \mathbf{x}_0)$  passing through  $\mathbf{x}_0 = (u_0, v_{10}, v_{20})$  at  $\tau = 0$ , satisfies  $\|\mathbf{x}(\tau, \mathbf{x}_0)\| < \epsilon$ , for  $\tau$  arbitrarily large. In this way we prove the stability (but not asymptotic stability) of  $J$ . ■

Hence, the super-accelerating cosmological solution represented by the singular point  $J$  is such that nearby solutions remain close to it, but do not approach it asymptotically.

#### 4.2.4.3 Analysis at infinity

Performing the transformation (4.33), the system (4.67) becomes

$$\begin{aligned}
\rho' = & -\frac{3}{2}\rho^3 \sin^4 \psi + \cos(2\theta) \left( \frac{3}{2}\rho (\rho^2 - 1) \sin^2 \psi + \right. \\
& \left. -\frac{1}{2}\mu_2 \rho^3 \cos \theta \cos \psi \sin^3 \psi \right) + \\
& + \cos \theta \left( -\lambda_1 \rho^2 \sqrt{6 - 6\rho^2} \sin^2 \theta \sin^3 \psi + \right. \\
& \left. -\frac{1}{4}\rho (-\mu_2 \rho^2 + \cos(2\psi)\rho^2 + \rho^2 + \mu_2) \sin(2\psi) \right), \quad (4.77)
\end{aligned}$$

$$\begin{aligned}
\theta' = & 3 \cos \theta \sin \theta + \frac{1}{2}\mu_2 \cot \psi \sin \theta + \\
& + \cos(2\theta) \left( \frac{\mu_2 \rho^2 \cos \psi}{2 - 2\rho^2} - \frac{\sqrt{\frac{3}{2}}\lambda_1 \rho}{\sqrt{1 - \rho^2}} \right) \sin \psi \sin \theta, \quad (4.78)
\end{aligned}$$

$$\begin{aligned}
\psi' = & \frac{3\rho^2 \cos \psi \sin^3 \psi}{2(\rho^2 - 1)} + \cos(2\theta) \left( \frac{\mu_2 \rho^2 \cos \theta \cos^2 \psi \sin^2 \psi}{2\rho^2 - 2} \right. \\
& \left. -\frac{3}{2} \cos \psi \sin \psi \right) + \\
& + \cos \theta \left( \frac{1}{2} \cos^2 \psi \left( -\frac{2\rho^2 \sin^2 \psi}{\rho^2 - 1} - \mu_2 \right) \right. \\
& \left. -\frac{\sqrt{6}\lambda_1 \rho \cos \psi \sin^2 \theta \sin^2 \psi}{\sqrt{1 - \rho^2}} \right). \quad (4.79)
\end{aligned}$$

Table 4.11: Asymptotic singular points of the system (4.67) (case 4) and their stability. We use the notations  $\beta = \frac{3}{\sqrt{13-4\mu_2+\mu_2^2}}$ ,  $\epsilon = \text{sign}(\mu_2)$  and  $\eta = \text{sign}(-11 - 4\mu_2 + \mu_2^2)$ . NH stands for nonhyperbolic.

Cr. P	Coordinates $\theta, \psi, x_r, y_r, z_r$	Eigenvalues	$\rho'$	Stability
$Q_{22}$	$0, 0, 0, 0, 1$	$0, 0$	$0$	NH; 3D center manifold
$Q_{23}$	$0, \frac{\pi}{2}, 1, 0, 0$	$0, +\infty$	$-\frac{3}{2}$	NH; 2D unstable manifold
$Q_{24}$	$\pi, \frac{\pi}{2}, -1, 0, 0$	$0, +\infty$	$-\frac{3}{2}$	NH; 2D unstable manifold
$Q_{25}$	$\frac{\pi}{4}, \frac{\pi}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0$	$0, +\infty$	$-\frac{3}{2}$	NH; 2D unstable manifold
$Q_{26}$	$\frac{3\pi}{4}, \frac{\pi}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0$	$0, +\infty$	$-\frac{3}{2}$	NH; 2D unstable manifold
$Q_{27}$	$\frac{\pi}{4}, \cos^{-1}\left(\frac{3}{\sqrt{11}}\right), \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}$	$-\epsilon\infty, -\infty$	$> 0$ if $\lambda_1 < -11$ $< 0$ if $\lambda_1 > -11$	sink if $\lambda_1 < -11$ and $\mu_2 > 0$ saddle otherwise
$Q_{28}$	$0, \cos^{-1}(\beta), \frac{\beta \mu_2-2 }{3}, 0, \beta$	$\epsilon\infty, -\infty$ if $\mu_2 < 2$ $+\infty, \eta\infty$ if $\mu_2 > 2$	$< 0$	source, if $\mu_2 > 2 + \sqrt{15}$ , saddle, otherwise
$Q_{29}$	$\pi, \cos^{-1}(\beta), -\frac{\beta \mu_2-2 }{3}, 0, \beta$	$-\epsilon\infty, \eta\infty$ if $\mu_2 < 2$ $-\infty, -\infty$ if $\mu_2 > 2$	$< 0$ if $\mu_2 < 2 - \sqrt[3]{18}$ or $\mu_2 > 2 + \sqrt[3]{18}$ $> 0$ if $2 < \mu_2 < 2 + \sqrt[3]{18}$ or $2 - \sqrt[3]{18} < \mu_2 < 2$	source, if $\mu_2 < 2 - \sqrt{15}$ sink, if $0 < \mu_2 < 2$ or $2 < \mu_2 < 2 + \sqrt[3]{18}$ ; saddle, otherwise

In the limit  $\rho \rightarrow 1$ , the leading terms in (4.77)-(4.79) are

$$\rho' \rightarrow -\frac{1}{4} \sin \psi \left( 6 \sin^3 \psi + \cos \theta \left( 4 \cos^3 \psi + \mu_2 \cos(2\theta) \sin \psi \sin(2\psi) \right) \right), \quad (4.80)$$

$$\theta' \rightarrow \frac{\mu_2 \cos(2\theta) \cos \psi \sin \theta \sin \psi}{2(1 - \rho^2)}, \quad (4.81)$$

$$\psi' \rightarrow -\frac{\cos \psi \sin^2 \psi (\cos \theta (\mu_2 \cos(2\theta) - 2) \cos \psi + 3 \sin \psi)}{2(1 - \rho^2)}. \quad (4.82)$$

The radial equation does not contain the radial coordinate, so the singular points can be obtained using the angular equations only. Setting  $\theta' = 0, \psi' = 0$ , we obtain the singular points which are listed in table 4.11. The stability of these points is studied by analyzing first the stability of the angular coordinates and then deducing, from the sign of equation (4.80), the stability on the radial direction.

In table 4.12 are displayed the values of the basic observables (4.6), (4.7), (4.8) and (4.9) for the singular points of the system (4.67) (case 4) as well as the solution types.

Table 4.12: Basic observables for the singular points of the system (4.67) (case 4). Solution types. We use the notations  $M(\mu_2) = -\frac{1}{9}(\mu_2 - 2)^2$  and  $N(\mu_2) = \frac{1}{6}(-\mu_2^2 + 4\mu_2 - 1)$ .

Cr. P	$\Omega_\phi$	$w_\phi$	$w_{\text{tot}}$	$q$	Solution type
$Q_{22}$	0	1	0	$\frac{1}{2}$	dust-like
$Q_{23}$	$-\infty$	1	$-\infty$	$-\infty$	(unphys.) big-rip
$Q_{24}$	$-\infty$	1	$-\infty$	$-\infty$	(unphys.) big-rip
$Q_{25}$	0	$-\infty$	$-\infty$	$-\infty$	big-rip
$Q_{26}$	0	$+\infty$	$-\infty$	$-\infty$	big-rip
$Q_{27}$	0	Indet.	$-\frac{2}{9}$	$\frac{1}{6}$	Matter-dominated
$Q_{28}$	$M(\mu_2)$	1	$M(\mu_2)$	$N(\mu_2)$	unphysical
$Q_{29}$	$M(\mu_2)$	1	$M(\mu_2)$	$N(\mu_2)$	unphysical

In order to perform the numerical experiments for the system (4.77)-(4.79) it is useful to rewrite the system in the cartesian coordinates  $x_r, y_r, z_r$ . The system (4.77)-(4.79) becomes

$$\begin{aligned} x'_r = & \frac{1}{2} \left( (6y_r^2 - 3z_r^2 + 3) x_r^3 + z_r (-2\mu_2 y_r^2 - (\mu_2 - 2) z_r^2 + \mu_2) x_r^2 + \right. \\ & + 3 (2y_r^4 + (z_r^2 - 1) y_r^2 + z_r^2 - 1) x_r + \sqrt{6} w \lambda_1 (2x_r^2 - 1) y_r^2 + \\ & \left. + \mu_2 z_r (2y_r^2 + z_r^2 - 1) \right), \end{aligned}$$



$$\begin{aligned}
y'_r &= \frac{1}{2}y_r \left( -(\mu_2 - 2)x_r z_r^3 - 3(x_r^2 - y_r^2 + 1)z_r^2 + \mu_2 x_r(1 - 2y_r^2)z_r + \right. \\
&\quad \left. + (2y_r^2 - 1)\left(\sqrt{6}w\lambda_1 x_r + 3(x_r^2 + y_r^2 - 1)\right) \right), \\
z'_r &= \frac{1}{2}z_r \left( (6y_r^2 - 3z_r^2 + 3)x_r^2 + \right. \\
&\quad \left. + \left( 2y_r^2 \left( \sqrt{6}w\lambda_1 - \mu_2 z_r \right) - (\mu_2 - 2)z_r(z_r^2 - 1) \right) x_r + \right. \\
&\quad \left. + 3y_r^2(2y_r^2 + z_r^2 - 1) \right) \\
w' &= \frac{1}{2}w \left( (6y_r^2 - 3z_r^2 + 3)x_r^2 + \right. \\
&\quad \left. + \left( -(\mu_2 - 2)z_r^3 + \mu_2 z_r + 2y_r^2 \left( \sqrt{6}w\lambda_1 - \mu_2 z_r \right) \right) x_r + \right. \\
&\quad \left. + 3y_r^2(2y_r^2 + z_r^2 - 1) \right)
\end{aligned} \tag{4.83}$$

where we have used the time re-scaling

$$d\tau \rightarrow \frac{d\tau}{1 - \rho^2}$$

which leave invariant the orbits of the phase-space and the time direction (see theorem 4) and we have introduced the auxiliary variable  $w = \sqrt{1 - x_r^2 - y_r^2 - z_r^2}$  to avoid that the numerical procedure becomes complex-valued at the singular points.

In order to examine the stability of  $Q_{22}$  we introduce the linear coordinate transformation

$$u_1 = \frac{x_r}{\mu_2}, \quad u_2 = y_r, \quad u_3 = z_r - 1, \quad u_4 = w$$

and Taylor expanding up to third order the system (4.83) becomes

$$\begin{aligned}
u'_1 &= \mu_2 u_1^2 + 3u_3 u_1 + u_2^2 + \frac{3u_3^2}{2} + u_3 + \mathcal{O}(3), \\
u'_2 &= u_1 u_2 \mu_2 - 3u_2 u_3 + \mathcal{O}(3), \\
u'_3 &= 2\mu_2(2 - \mu_2)u_1 u_3 + \mathcal{O}(3), \\
u'_4 &= u_1 u_4 \mu_2 + \mathcal{O}(3).
\end{aligned} \tag{4.84}$$

The linear part of the vector field (4.84) is given by

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By computing the action of  $\mathbf{L}_J^{(2)}$  on each basis element on  $H^2$  (the vector space of

4-dimensional vector fields of second order) we have

$$\begin{aligned}
\mathbf{L}_J^{(2)}(H^2) = & \text{span} \left\{ \begin{pmatrix} u_1^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 u_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 u_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2 u_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \right. \\
& \begin{pmatrix} u_3^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 u_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2 u_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_3 u_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_4^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
& \begin{pmatrix} 0 \\ u_1 u_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2 u_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2 u_4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1 u_2 \\ 0 \end{pmatrix}, \\
& \begin{pmatrix} 0 \\ 0 \\ u_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_2 u_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_2 u_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_1 u_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_2^2 \end{pmatrix}, \\
& \left. \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_2 u_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_2 u_4 \end{pmatrix} \right\}. \tag{4.85}
\end{aligned}$$

Thus, the second order terms that are linear combinations of the twenty two vectors in (4.85) can be eliminated [358]. To determine the nature of the second order terms that

cannot be eliminated we must compute the complementary space of (4.85) which is

$$\begin{aligned}
 G^2 = & \text{span} \left\{ \begin{pmatrix} 0 \\ u_1^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1 u_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_3^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1 u_4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_3 u_4 \\ 0 \\ 0 \end{pmatrix}, \right. \\
 & \begin{pmatrix} 0 \\ u_4^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1 u_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_3^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1 u_4 \\ 0 \end{pmatrix}, \\
 & \begin{pmatrix} 0 \\ 0 \\ u_3 u_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_4^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_1 u_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_3^2 \end{pmatrix}, \\
 & \left. \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_1 u_4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_3 u_4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_4^2 \end{pmatrix} \right\}. \tag{4.86}
 \end{aligned}$$

The normal form of (4.84) is given by

$$\begin{aligned}
 v_1' &= v_3 + \mathcal{O}(3), \\
 v_2' &= \mathcal{O}(3), \\
 v_3' &= \mu_2(2 - \mu_2)v_1 v_3 + \mathcal{O}(3), \\
 v_4' &= v_1 v_4 \mu_2 + \mathcal{O}(3). \tag{4.87}
 \end{aligned}$$

The general solution of (4.87) is given by

$$\begin{aligned}
 v_1 &= \frac{\sqrt{2}\sqrt{c_1} \tanh\left(\frac{\sqrt{c_1}\sqrt{\mu_2-2}\sqrt{\mu_2}(\tau+2c_1)}{\sqrt{2}}\right)}{\sqrt{\mu_2-2}\sqrt{\mu_2}}, \\
 v_3 &= c_2 \text{sech}^2\left(\frac{\sqrt{c_1}\sqrt{\mu_2-2}\sqrt{\mu_2}(\tau+2c_1)}{\sqrt{2}}\right), \\
 v_4 &= c_3 \cosh^{\frac{2}{\mu_2-2}}\left(\frac{\sqrt{c_1}\sqrt{\mu_2-2}\sqrt{\mu_2}(\tau+2c_1)}{\sqrt{2}}\right). \tag{4.88}
 \end{aligned}$$

By analyzing the qualitative behavior of solutions (4.88) we obtain that the dynamical character  $Q_{22}$  is very sensible to the changes on the initial conditions. Thus, we argue that  $Q_{22}$  is unstable.

By an explicit calculation we find that the center manifolds of  $Q_{23,24}$  are  $x_r = \pm \left(1 - \frac{y_r^2}{2}\right) + \mathcal{O}(y_r)^3 = \pm \sqrt{1 + y^2}$ ,  $z_r = w = 0$ . The dynamics on the center manifolds of  $Q_{15,16}$  are given by  $u' = \frac{3}{2}u^2 + \mathcal{O}(u)^5$ , where  $u = y_r$ . From this follows that  $Q_{15,16}$  are local sources since their center manifolds are unstable (the origin is a degenerate local maximum of the potential function  $U(u) = -\frac{3u^4}{8}$ ).

We obtain by an explicit calculation that the center manifold of  $Q_{25,26}$  is given up to fourth order by the graph  $x_r \pm y_r \mp \sqrt{2} = \mp \sqrt{2}u^2$ ,  $z_r = 0$ ,  $w = 0$  where  $u \equiv y_r - \frac{\sqrt{2}}{2}$ . The dynamics on the center manifold is given by

$$u' = -6u^3 + \mathcal{O}(5).$$

From this follows that the center manifold is stable since the origin is a degenerate minimum of the potential  $U(u) = \frac{3u^4}{2}$ . Hence,  $Q_{25,26}$  are of saddle type.

In the figures 4.11 and 4.12 are displayed some orbits in the Poincaré (global) phase-space of Model 4, for the parameter values  $\lambda_1 = 1.0$  and  $\mu_2 = 2.1 + \sqrt{15}$ . In the figures the attractor in the finite region is  $J$ . The points at infinity  $Q_{25,26}$  are the local sources whereas  $Q_{27,28,29}$  are saddles.

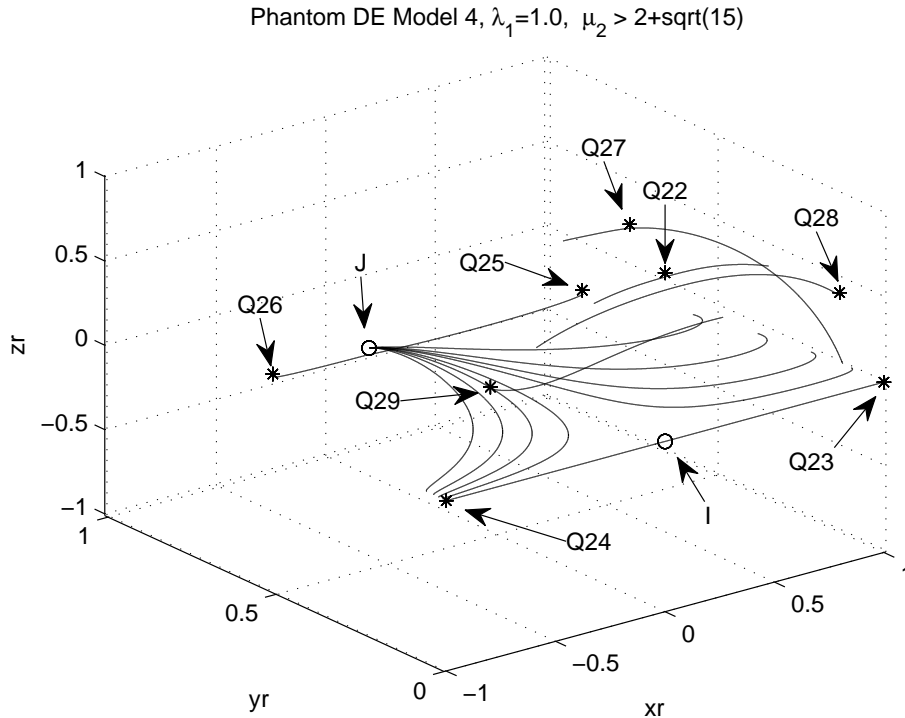


Figure 4.11: Poincaré (global) phase-space of Model 4, for the parameter values  $\lambda_1 = 1.0$  and  $\mu_2 = 2.1 + \sqrt{15}$ . In the figure the attractor in the finite region is  $J$ . The points at infinity  $Q_{25,26}$  are the local sources whereas  $Q_{27,28,29}$  are saddles.

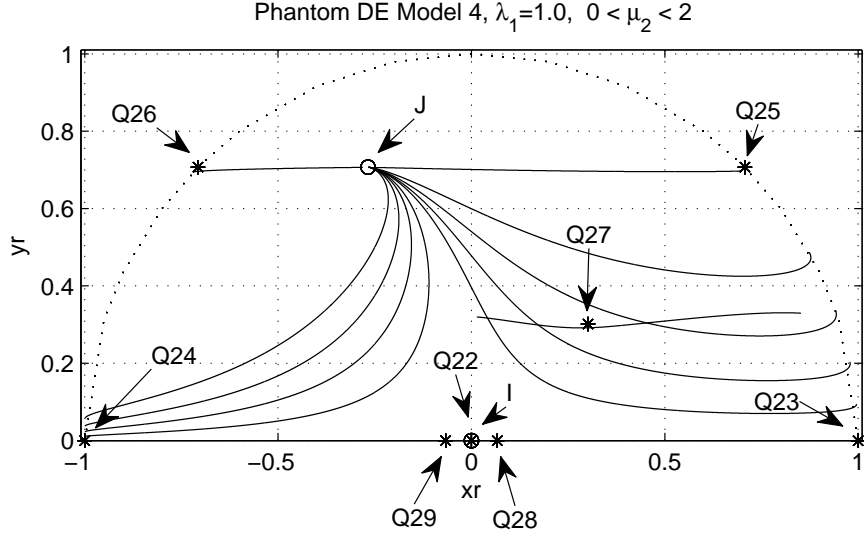


Figure 4.12: Projection of 4.11 on the plane  $x_r$ - $y_r$ .

#### 4.2.4.4 Cosmological implications and discussion: Model 4

In this case, the critical points I and J exist always. The point I corresponds to a flat, non-accelerating, matter-dominated universe. J corresponds to a dark-energy dominated universe, that super-accelerates [433]. Similarly to the previous cases, the stable manifolds of I and J are 1D or 2D respectively, and thus almost all orbits of the cosmological system cannot be attracted by I at late times. We proved, however, that J is always stable (but not asymptotically stable) having a large chance to be the late-time attractor. Since they cannot lead to  $0 < \Omega_\phi < 1$ , they are not relevant to solve the Coincidence Problem. Finally, by using Poincaré projection, we have obtained that the physical solutions at infinity are  $Q_{22,25,26,27}$ .  $Q_{22}$  represents a matter-dominated solution with effective equation of state  $w_{\text{tot}} = 0$  (dust-like) which is unstable by our previous analysis.  $Q_{25}$  and  $Q_{26}$  corresponds to initial big-rip singularities due that  $q \rightarrow -\infty$  and  $w_{\text{tot}} \rightarrow -\infty$ . That is, the points at infinity represents super-accelerating ( $q \ll 0$ ) phantom solutions ( $w_{\text{tot}} \ll -1$ ). The unphysical solutions are  $Q_{23}$  and  $Q_{24}$  that represents unphysical big-rip singularities since  $q \rightarrow -\infty$  and  $w_{\text{tot}} \rightarrow -\infty$ , and  $Q_{28,29}$ . The last two singular points satisfy  $\Omega_\phi < 0$  for  $\mu_2 \neq 2$ . In the case  $\mu_2 = 2$  they reduce to  $Q_{22}$ . None of these solutions allows to solve the coincidence problem.

Therefore, an exponential potential and a power-law-dependent dark-matter particle mass, cannot solve the coincidence problem.

## 4.3 Conclusions

In this chapter we have investigated the phantom cosmological scenario, with varying-mass dark-matter particles due to the interaction between dark-matter and dark-energy sectors. In particular, we performed a detailed phase-space analysis of various models, with either exponentially or power-law dependent dark-matter particle mass, in exponential or power-law scalar field potentials in both finite and infinite regions. These functions cover a wide range of the possible forms, and they correspond to the cases that can accept a reasonable theoretical justification [193, 206, 425, 426, 428]. In each case we extracted the critical points in both finite and infinite regions, we determined their stability, and we calculated the basic cosmological observables, namely the total equation-of-state parameter  $w_{tot}$  and  $\Omega_{DE}$  (attributed to the phantom field). Our basic goal was to examine whether there exist late-time attractors, corresponding to accelerating universe and possessing  $\Omega_{DE}/\Omega_{DM} \approx \mathcal{O}(1)$ , thus satisfying the basic observational requirements.

The new results of this investigation are the following:

1. By performing a Poncaré projection we find that for exponential potential and exponentially-dependent dark-matter particle mass there is no late-time attractors in the infinite region. Thus, following the discussion in the section 4.2.1.1 the relevant late-time attractor with physical sense is the phantom-dominated super-accelerated solution,  $A$ , for the choice of parameters in the range  $\lambda_1 (\mu_1 - \lambda_1) < 3$ .
2. We have proved, using the Center Manifold theory the proposition 16 that states that for  $\lambda_2 < 0$ , the de Sitter solution for Power-law potential and power-law dependent dark-matter particle mass is locally asymptotically stable. For  $\lambda_2 > 0$ , it is locally unstable (saddle type). For  $\lambda_2 = 0$ , it is stable but not asymptotically stable.
3. By investigating the global phase space of the above models through Poincaré projection, we have obtained saddle points corresponding to matter-dominated solutions with effective dust-like equation of state; super-accelerating early-time phantom solutions and unphysical big-rip singularities.
4. For power-law potential and exponentially-dependent dark-matter particle mass we have proved proposition 17 which states that for  $\lambda_2 < 0$  the de Sitter solution is locally asymptotically stable. For  $\lambda_2 > 0$  it is locally unstable (saddle type) whereas for  $\lambda_2 = 0$  it is stable but not asymptotically stable.
5. By investigating the global phase space dynamics through Poincaré projection of the above model-class, we have obtained unstable matter-dominated solution with dust-like effective equation of state; super-accelerating early-time phantom solutions and unphysical big-rip singularities. None of these solutions allows to solve the coincidence problem.

6. For exponential potential and power-law-dependent dark-matter particle mass, we have proved the proposition 18 that states that the phantom-dominated solution at finite region is stable but is not asymptotically stable.
7. By investigating the global phase space dynamics through Poincaré projection of the above model-class, we have obtained saddle points corresponding to matter-dominated solutions with effective dust-like equation of state; super-accelerating early-time phantom solutions and unphysical big-rip singularities.

Thus, we have that for power-law potential and either power-law- or exponentially-dependent dark matter particles mass, the de Sitter solution is locally asymptotically stable provided  $\lambda_2 < 0$  and stable, but not asymptotically stable for  $\lambda_2 = 0$ . These results suggest that the dynamical character of the de Sitter solution depends more on the potential of the scalar field rather than on the mass-varying function. We have obtained also that for the case of an exponential potential with an exponentially-dependent dark-matter particle mass, the cosmological system possesses a relevant late-time (phantom) attractor. For the exponential potential and power-law-dependent dark-matter particle mass, the phantom solution is a relevant late-time attractor. However, in all the examined cases, solutions having  $\Omega_{\text{DE}}/\Omega_{\text{DM}} \approx \mathcal{O}(1)$  are not relevant attractors at late times. By using the Poicaré projection method we have investigate the infinity region obtaining as interesting result the existence of both early-time phantom solutions and past big-rip singularities.

Therefore, summarizing, the coincidence problem cannot be solved or even alleviated in varying-mass dark matter particles models in the framework of phantom cosmology, in a radical contrast with the corresponding quintessence case [193, 206, 426]. This conclusion agrees with that of [421], that interacting phantom cosmology cannot solve the coincidence problem. It seems that interacting phantom cosmology, either directly or through the dependence of the dark-matter particle mass, cannot fulfill the basic requirements that led to its construction, that is to provide stable accelerating late-time solutions which can solve the coincidence problem. An alternative direction could be to consider a specially constructed potential or dark-matter particle mass in order to solve the coincidence problem, but this would imply significant loss of simplicity, generality, and theoretical justification of the model.

The aforementioned conclusion has been extracted by the negative-kinetic-energy realization of phantom, which does not cover the whole class of phantom models. However, since it is a qualitative statement it should intuitively be robust for general phantom scenarios, too. Therefore, phantom cosmology with varying-mass dark matter particles cannot easily act as a successful candidate to describe dark energy.





# Chapter 5

## Phase-space analysis of Hořava-Lifshitz cosmology

In this chapter we perform a detailed phase-space analysis of Hořava-Lifshitz cosmology, with and without the detailed-balance condition. Under detailed-balance we find that the universe can reach a bouncing-oscillatory state at late times, in which dark-energy, behaving as a simple cosmological constant, is dominant. In the case where the detailed-balance condition is relaxed, we find that the universe reaches an eternally expanding, dark-energy-dominated solution, with the oscillatory state preserving also a small probability. Although this analysis indicates that Hořava-Lifshitz cosmology can be compatible with observations, it does not enlighten the discussion about its possible conceptual and theoretical problems.

### 5.1 Introduction

Two years ago Hořava proposed a power-counting renormalizable theory with consistent ultra-violet (UV) behavior [134]. Although presenting an infrared (IR) fixed point, namely General Relativity, in the UV the theory exhibits an anisotropic, Lifshitz scaling between time and space. Due to these novel features, there has been a large amount of effort in examining the properties of the theory [248, 249, 254, 263]. Furthermore, application of Hořava-Lifshitz gravity as a cosmological framework gives rise to Hořava-Lifshitz cosmology, which proves to lead to interesting behavior [265]. In particular, one can examine specific solution subclasses [266], the phase-space behavior [151], the gravitational wave production [279], the perturbation spectrum [272], the matter bounce [282], the black hole properties [285], the dark energy phenomenology [295], the observational constraints on the parameters of the theory [435], the astrophysical phenomenology [299], the thermodynamic properties [289] etc. However, despite this extended research, there are still many ambiguities if Hořava-Lifshitz gravity is reliable and capable of a successful

description of the gravitational background of our world, as well as of the cosmological behavior of the universe [263, 254].

Let us briefly review the scenario where the cosmological evolution is governed by the simple version of Hořava-Lifshitz gravity [265]. The dynamical variables are the lapse and shift functions,  $N$  and  $N_i$  respectively, and the spatial metric  $g_{ij}$  (roman letters indicate spatial indices). In terms of these fields the full metric is written as:

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (5.1)$$

and the scaling transformation of the coordinates reads:  $t \rightarrow l^3 t$  and  $x^i \rightarrow l x^i$ .

### 5.1.1 Detailed Balance

The gravitational action is decomposed into a kinetic and a potential part as  $S_g = \int dt d^3x \sqrt{g} N (\mathcal{L}_K + \mathcal{L}_V)$ . The assumption of detailed balance [134] reduces the possible terms in the Lagrangian, and it allows for a quantum inheritance principle, since the  $(D+1)$ -dimensional theory acquires the renormalization properties of the  $D$ -dimensional one. Under the detailed balance condition the full action of Hořava-Lifshitz gravity is given by

$$\begin{aligned} S_g = & \int dt d^3x \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) \right. \\ & + \frac{\kappa^2}{2w^4} C_{ij} C^{ij} - \frac{\kappa^2 \mu}{2w^2} \frac{\epsilon^{ijk}}{\sqrt{g}} R_{il} \nabla_j R_k^l + \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} \\ & \left. - \frac{\kappa^2 \mu^2}{8(3\lambda - 1)} \left[ \frac{1 - 4\lambda}{4} R^2 + \Lambda R - 3\Lambda^2 \right] \right\}, \quad (5.2) \end{aligned}$$

where  $K_{ij} = (g_{ij} - \nabla_i N_j - \nabla_j N_i) / 2N$  is the extrinsic curvature and  $C^{ij} = \epsilon^{ijk} \nabla_k (R_i^j - R \delta_i^j / 4) / \sqrt{g}$  the Cotton tensor, and the covariant derivatives are defined with respect to the spatial metric  $g_{ij}$ .  $\epsilon^{ijk}$  is the totally antisymmetric unit tensor,  $\lambda$  is a dimensionless constant and the variables  $\kappa$ ,  $w$  and  $\mu$  are constants. Finally, we mention that in action (5.2) we have already performed the usual analytic continuation of the parameters  $\mu$  and  $w$  of the original version of Hořava-Lifshitz gravity, since such a procedure is required in order to obtain a realistic cosmology [266, 297].

In order to add the matter component we follow the hydrodynamical approach of adding a cosmological stress-energy tensor to the gravitational field equations, by demanding to recover the usual general relativity formulation in the low-energy limit [436]. Thus, this matter-tensor is a hydrodynamical approximation with  $\rho_m$  and  $p_m$  (or  $\rho_m$  and  $w_m$ ) as parameters. Similarly, one can additionally include the standard-model-radiation component, with the additional parameters  $\rho_r$  and  $w_r$ .

In order to investigate cosmological frameworks, we impose the projectability condi-

tion [263] and we use an FRW metric

$$N = 1, \quad g_{ij} = a^2(t)\gamma_{ij}, \quad N^i = 0, \quad (5.3)$$

with

$$\gamma_{ij}dx^i dx^j = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega_2^2, \quad (5.4)$$

where  $K = -1, 0, +1$  corresponding to open, flat, and closed universe respectively. By varying  $N$  and  $g_{ij}$ , we extract the Friedmann equations:

$$\begin{aligned} H^2 &= \frac{\kappa^2}{6(3\lambda - 1)} (\rho_m + \rho_r) \\ &+ \frac{\kappa^2}{6(3\lambda - 1)} \left[ \frac{3\kappa^2 \mu^2 K^2}{8(3\lambda - 1)a^4} + \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)} \right] \\ &- \frac{\kappa^4 \mu^2 \Lambda K}{8(3\lambda - 1)^2 a^2}, \end{aligned} \quad (5.5)$$

$$\begin{aligned} \dot{H} + \frac{3}{2}H^2 &= -\frac{\kappa^2}{4(3\lambda - 1)} (w_m \rho_m + w_r \rho_r) \\ &- \frac{\kappa^2}{4(3\lambda - 1)} \left[ \frac{\kappa^2 \mu^2 K^2}{8(3\lambda - 1)a^4} - \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)} \right] \\ &- \frac{\kappa^4 \mu^2 \Lambda K}{16(3\lambda - 1)^2 a^2}, \end{aligned} \quad (5.6)$$

where  $H \equiv \frac{\dot{a}}{a}$  is the Hubble parameter. As usual,  $\rho_m$  follows the standard evolution equation  $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$ , while  $\rho_r$  follows  $\dot{\rho}_r + 3H(\rho_r + p_r) = 0$ . Finally, concerning the dark-energy sector we can define

$$\rho_{DE} \equiv \frac{3\kappa^2 \mu^2 K^2}{8(3\lambda - 1)a^4} + \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)} \quad (5.7)$$

$$p_{DE} \equiv \frac{\kappa^2 \mu^2 K^2}{8(3\lambda - 1)a^4} - \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)}. \quad (5.8)$$

The term proportional to  $a^{-4}$  is the usual “dark radiation term”, present in Hořava-Lifshitz cosmology [265], while the constant term is just the explicit cosmological constant. Therefore, in expressions (5.7),(5.8) we have defined the energy density and pressure for the effective dark energy, which incorporates the aforementioned contributions. Note that using (5.7),(5.8) it is straightforward to show that these dark energy quantities satisfy the standard evolution equation:  $\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0$ .

If we require expressions (5.5) to coincide with the standard Friedmann equations, in

units where  $c = 1$  we set [265]:  $G = \frac{\kappa^2}{16\pi(3\lambda-1)}$  and  $\frac{\kappa^4\mu^2\Lambda}{8(3\lambda-1)^2} = 1$  where  $G$  is the Newton's constant.

### 5.1.2 Beyond Detailed Balance

The aforementioned formulation of Hořava-Lifshitz cosmology has been performed under the imposition of the detailed-balance condition. However, in the literature there is a discussion whether this condition leads to reliable results or if it is able to reveal the full information of Hořava-Lifshitz gravity [265]. Therefore, one needs to investigate also the Friedman equations in the case where detailed balance is relaxed. In such a case one can in general write [151, 263]:

$$\begin{aligned} H^2 &= \frac{2\sigma_0}{(3\lambda-1)}(\rho_m + \rho_r) \\ &+ \frac{2}{(3\lambda-1)}\left[\frac{\sigma_1}{6} + \frac{\sigma_3 K^2}{6a^4} + \frac{\sigma_4 K}{6a^6}\right] \\ &+ \frac{\sigma_2}{3(3\lambda-1)}\frac{K}{a^2} \end{aligned} \quad (5.9)$$

$$\begin{aligned} \dot{H} + \frac{3}{2}H^2 &= -\frac{3\sigma_0}{(3\lambda-1)}(w_m\rho_m + w_r\rho_r) \\ &- \frac{3}{(3\lambda-1)}\left[-\frac{\sigma_1}{6} + \frac{\sigma_3 K^2}{18a^4} + \frac{\sigma_4 K}{6a^6}\right] \\ &+ \frac{\sigma_2}{6(3\lambda-1)}\frac{K}{a^2}, \end{aligned} \quad (5.10)$$

where  $\sigma_0 \equiv \kappa^2/12$ , and the constants  $\sigma_i$  are arbitrary (with  $\sigma_2$  being negative and  $\sigma_4$  positive). Furthermore, the dark-energy quantities are generalized to

$$\rho_{DE}|_{\text{non-db}} \equiv \frac{\sigma_1}{6} + \frac{\sigma_3 K^2}{6a^4} + \frac{\sigma_4 K}{6a^6} \quad (5.11)$$

$$p_{DE}|_{\text{non-db}} \equiv -\frac{\sigma_1}{6} + \frac{\sigma_3 K^2}{18a^4} + \frac{\sigma_4 K}{6a^6}. \quad (5.12)$$

Again, it is easy to show that

$$\dot{\rho}_{DE}|_{\text{non-db}} + 3H(\rho_{DE}|_{\text{non-db}} + p_{DE}|_{\text{non-db}}) = 0. \quad (5.13)$$

Finally, if we force (5.9),(5.10) to coincide with the standard Friedmann equations, we obtain:  $G = \frac{6\sigma_0}{8\pi(3\lambda-1)}$  and  $\sigma_2 = -3(3\lambda-1)$ .

The above basic models of Hořava-Lifshitz cosmology proves to have very interesting cosmological behavior [151, 265, 266, 272, 279, 282, 285, 289, 295, 299, 435]. However, the gravitational sector itself proves to have instabilities that cannot be cured by

simple tricks such as analytic continuation [254, 263]. Therefore, it is necessary to try to construct suitable extensions that are free of such problems.

A quite general power-counting renormalizable action is [437]:

$$S = S_{kin} + S_1 + S_2 + S_{new}, \quad (5.14)$$

with

$$\begin{aligned} S_{kin} &= \alpha \int dt d^3x \sqrt{g} N [(K_{ij} K^{ij} - l K^2)] \\ S_1 &= \int dt d^3x \sqrt{g} N \left[ \gamma_0 \frac{\mathbf{e}^{ijk}}{\sqrt{g}} R_{il} \nabla_j R^l_k + \zeta R_{ij} R^{ij} + \eta R^2 + \xi R + \sigma \right] \\ S_2 &= \int dt d^3x \sqrt{g} N [\beta_0 C_{ij} C^{ij} + \beta_1 R \square R + \beta_2 R^3 \\ &\quad + \beta_3 R R_{ij} R^{ij} + \beta_4 R_{ij} R^{ik} R^j_k] \\ S_{new} &= \int dt d^3x \sqrt{g} N [a_1 (a_i a^i) + a_2 (a_i a^i)^2 + a_3 R^{ij} a_i a_j \\ &\quad + a_4 R \nabla_i a^i + a_5 \nabla_i a_j \nabla^i a^j + a_6 \nabla^i a_i (a_j a^j) + \dots]. \end{aligned} \quad (5.15)$$

Thus, apart from the known kinetic, detailed-balance and beyond-detailed-balance combinations that constitute the Hořava-Lifshitz gravitational action, in (5.15) we have added a new combination, based on the term [438]:

$$a_i \equiv \frac{\partial_i N}{N}, \quad (5.16)$$

which breaks the projectability condition, and the ellipsis in (5.15) refers to dimension six terms involving  $a_i$  as well as curvatures.

Such a new combination of terms seems to alleviate the problems of Hořava-Lifshitz gravity, although there could still be some ambiguities. Therefore, one should repeat all the relevant investigations of the literature for this extended version of the theory.

## 5.2 The cosmological equations

The cosmological equations of Hořava-Lifshitz cosmology including a scalar field matter source, under the imposition of the detailed-balance condition, are:

$$\begin{aligned}
H^2 &= \frac{\kappa^2}{6(3\lambda-1)} \left[ \frac{3\lambda-1}{4} \dot{\phi}^2 + V(\phi) \right] + \\
&+ \frac{\kappa^2}{6(3\lambda-1)} \left[ -\frac{3\kappa^2\mu^2k^2}{8(3\lambda-1)a^4} - \frac{3\kappa^2\mu^2\Lambda^2}{8(3\lambda-1)} \right] + \\
&+ \frac{\kappa^4\mu^2\Lambda k}{8(3\lambda-1)^2a^2}, \tag{5.17}
\end{aligned}$$

$$\begin{aligned}
\dot{H} + \frac{3}{2}H^2 &= -\frac{\kappa^2}{4(3\lambda-1)} \left[ \frac{3\lambda-1}{4} \dot{\phi}^2 - V(\phi) \right] - \\
&- \frac{\kappa^2}{4(3\lambda-1)} \left[ -\frac{\kappa^2\mu^2k^2}{8(3\lambda-1)a^4} + \frac{3\kappa^2\mu^2\Lambda^2}{8(3\lambda-1)} \right] + \\
&+ \frac{\kappa^4\mu^2\Lambda k}{16(3\lambda-1)^2a^2}, \tag{5.18}
\end{aligned}$$

where we have defined the Hubble parameter as  $H \equiv \frac{\dot{a}}{a}$ , and we have neglected radiation from the cosmological budget.

Finally, the equation of motion for the scalar field reads:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{2}{3\lambda-1} \frac{dV(\phi)}{d\phi} = 0. \tag{5.19}$$

At this stage we can define the energy density and pressure for the scalar field responsible for the matter content of the Hořava-Lifshitz universe:

$$\rho_M \equiv \rho_\phi = \frac{3\lambda-1}{4} \dot{\phi}^2 + V(\phi) \tag{5.20}$$

$$p_M \equiv p_\phi = \frac{3\lambda-1}{4} \dot{\phi}^2 - V(\phi). \tag{5.21}$$

Concerning the dark-energy sector we can define

$$\rho_{DE} \equiv -\frac{3\kappa^2\mu^2k^2}{8(3\lambda-1)a^4} - \frac{3\kappa^2\mu^2\Lambda^2}{8(3\lambda-1)} \tag{5.22}$$

$$p_{DE} \equiv -\frac{\kappa^2\mu^2k^2}{8(3\lambda-1)a^4} + \frac{3\kappa^2\mu^2\Lambda^2}{8(3\lambda-1)}. \tag{5.23}$$

The term proportional to  $a^{-4}$  is the usual “dark radiation term”, present in Hořava-Lifshitz cosmology [264, 265]. Finally, the constant term is just the explicit (negative) cosmological constant. Therefore, in expressions (5.22),(5.23) we have defined the energy density

and pressure for the effective dark energy, which incorporates the aforementioned contributions.

Using the above definitions, we can re-write the Friedmann equations (5.17),(5.18) in the standard form:

$$H^2 = \frac{\kappa^2}{6(3\lambda - 1)} [\rho_M + \rho_{DE}] + \frac{\beta k}{a^2} \quad (5.24)$$

$$\dot{H} + \frac{3}{2}H^2 = -\frac{\kappa^2}{4(3\lambda - 1)} [p_M + p_{DE}] + \frac{\beta k}{2a^2}. \quad (5.25)$$

In these relations we have defined  $\beta \equiv \frac{\kappa^4 \mu^2 \Lambda}{8(3\lambda - 1)^2}$ , which is the coefficient of the curvature term. Additionally, we could also define an effective Newton's constant and an effective light speed [264, 265], but we prefer to keep  $\frac{\kappa^2}{6(3\lambda - 1)}$  in the expressions, just to make clear the origin of these terms in Hořava-Lifshitz cosmology. Finally, note that using (5.19) it is straightforward to see that the aforementioned dark matter and dark energy quantities verify the standard evolution equations:

$$\dot{\rho}_M + 3H(\rho_M + p_M) = 0 \quad (5.26)$$

$$\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0. \quad (5.27)$$

In the literature there is a discussion whether the detailed-balance condition leads to reliable results or if it is able to reveal the full information of Hořava-Lifshitz gravity [264, 265]. Thus, for completeness, we add here the Friedmann equation in the case where detailed balance is relaxed. In such a case one can in general write [253, 254, 263]:

$$\begin{aligned} H^2 &= \frac{2\sigma_0}{(3\lambda - 1)} \left[ \frac{3\lambda - 1}{4} \dot{\phi}^2 + V(\phi) \right] + \\ &+ \frac{2}{(3\lambda - 1)} \left[ \frac{\sigma_1}{6} + \frac{\sigma_3 k^2}{6a^4} + \frac{\sigma_4 k}{a^6} \right] + \\ &+ \frac{\sigma_2}{3(3\lambda - 1)} \frac{k}{a^2} \end{aligned} \quad (5.28)$$

$$\begin{aligned} \dot{H} + \frac{3}{2}H^2 &= -\frac{3\sigma_0}{(3\lambda - 1)} \left[ \frac{3\lambda - 1}{4} \dot{\phi}^2 - V(\phi) \right] - \\ &- \frac{3}{(3\lambda - 1)} \left[ -\frac{\sigma_1}{6} + \frac{\sigma_3 k^2}{18a^4} + \frac{\sigma_4 k}{6a^6} \right] + \\ &+ \frac{\sigma_2}{6(3\lambda - 1)} \frac{k}{a^2}, \end{aligned} \quad (5.29)$$

where  $\sigma_0 \equiv \kappa^2/12$ , and the constants  $\sigma_i$  are arbitrary (although one can set  $\sigma_2$  to be positive too). Thus, the effect of the detailed-balance relaxation is the decoupling of the coefficients, together with the appearance of a term proportional to  $a^{-6}$ . This term

has a negligible impact at large scale factors, however it could play a significant role at small ones. Finally, in the non-detailed-balanced case, the energy density and pressure for matter coincide with those of detailed-balance scenario (expressions (5.20),(5.21)), since the detailed-balance condition affects only the gravitational sector of the theory and has nothing to do with the matter content of the universe. However, the corresponding quantities for dark energy are generalized to

$$\rho_{DE}|_{\text{non-db}} \equiv \frac{\sigma_1}{6} + \frac{\sigma_3 k^2}{6a^4} + \frac{\sigma_4 k}{a^6} \quad (5.30)$$

$$p_{DE}|_{\text{non-db}} \equiv -\frac{\sigma_1}{6} + \frac{\sigma_3 k^2}{18a^4} + \frac{\sigma_4 k}{6a^6}. \quad (5.31)$$

### 5.3 Detailed balance: Phase-space analysis

In order to perform the phase-space and stability analysis of the Hořava-Lifshitz universe, we have to transform the cosmological equations into an autonomous dynamical system [409, 429, 430, 431]. This will be achieved by introducing the auxiliary variables:

$$x = \frac{\kappa \dot{\phi}}{2\sqrt{6}H}, \quad (5.32)$$

$$y = \frac{\kappa \sqrt{V(\phi)}}{\sqrt{6}H\sqrt{3\lambda-1}} \quad (5.33)$$

$$z = \frac{\kappa^2 \mu}{4(3\lambda-1)a^2 H} \quad (5.34)$$

$$u = \frac{\kappa^2 \Lambda \mu}{4(3\lambda-1)H}, \quad (5.35)$$

together with  $M = \ln a$ . Thus, it is easy to see that for every quantity  $F$  we acquire  $\dot{F} = H \frac{dF}{dM}$ . Using these variables we can straightforwardly obtain the density parameters of dark matter and dark energy (through expressions (5.20), (5.22)) as:

$$\Omega_M \equiv \frac{\kappa^2}{6(3\lambda-1)H^2} \rho_M = x^2 + y^2, \quad (5.36)$$

$$\Omega_{DE} \equiv \frac{\kappa^2}{6(3\lambda-1)H^2} \rho_{DE} = -k^2 z^2 - u^2, \quad (5.37)$$

and in addition we can calculate the corresponding equation-of-state parameters:

$$w_M \equiv \frac{p_M}{\rho_M} = \frac{x^2 - y^2}{x^2 + y^2}, \quad (5.38)$$

$$w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}} = \frac{k^2 z^2 - 3u^2}{3k^2 z^2 + 3u^2}. \quad (5.39)$$



We mention that these relations are always valid, that is independently of the specific state of the system (they are valid in the whole phase-space and not only at the singular points). Finally, for completeness, and observing (5.24), we can define the curvature density parameter as:

$$\Omega_k \equiv \frac{\beta k}{H^2 a^2} = 2kuz. \quad (5.40)$$

Using the auxiliary variables (5.32),(5.33),(5.34),(5.35) the cosmological equations of motion (5.24), (5.25), (5.26) and (5.27), can be transformed into an autonomous form  $\mathbf{X}' = \mathbf{f}(\mathbf{X})$ , where  $\mathbf{X}$  is the column vector constituted by the auxiliary variables,  $\mathbf{f}(\mathbf{X})$  the corresponding column vector of the autonomous equations, and prime denotes derivative with respect to  $M = \ln a$ .

In the following we perform a phase-space analysis of the cosmological system at hand. As we can see from the Friedmann equations (5.17), (5.18) one can have a zero or non-zero cosmological constant, in a flat or non-flat universe. Thus, for simplicity we investigate separately the corresponding four cases. Finally, note that we assume  $\lambda > \frac{1}{3}$  as required by the consistency of the Hořava gravitational background, but we do not impose any other constraint on the model parameters (although one could do so using the light speed and Newton's constant values) in order to remain as general as possible.

### 5.3.1 Case 1: Flat universe with $\Lambda = 0$

In this scenario the variable  $u$  is irrelevant, and the Friedmann equations (5.17), (5.18) become:

$$1 = x^2 + y^2 \quad (5.41)$$

$$\frac{H'}{H} = -3x^2. \quad (5.42)$$

Thus, after using the first of these relations in order to eliminate one variable, the corresponding autonomous system writes:

$$x' = (3x - \sqrt{6}s)(x^2 - 1), \quad (5.43)$$

$$z' = (3x^2 - 2)z. \quad (5.44)$$

We mention that for simplicity we have set  $s = -\frac{1}{\kappa V(\phi)} \frac{dV(\phi)}{d\phi}$  and we have assumed it to be a constant, that is we are investigating the usual exponential potentials. However, as we will see this is not necessary, since the most important results of the present work are independent of the matter sector.

The autonomous system (5.43)-(5.44) is defined in the phase space

$$\Psi = \{(x, z) : -1 \leq x \leq 1, z \in \mathbb{R}\}.$$

As we observe, this phase plane is not compact since  $z$  is in general unbounded. However, the system is integrable and the orbit in the plane  $\Psi$ , passing initially through  $(x_0, z_0)$ , can be obtained explicitly and it is given by the graph

$$z(x) = z_0 \left( \frac{3x - \sqrt{6}s}{3x_0 - \sqrt{6}s} \right)^{1 + \frac{1}{2s^2 - 3}} \left( \frac{x^2 - 1}{x_0^2 - 1} \right)^{\frac{1}{6 - 4s^2}} \cdot \exp \left\{ \frac{\sqrt{6}s [\tanh^{-1}(x) - \tanh^{-1}(x_0)]}{6s^2 - 9} \right\}. \quad (5.45)$$

### 5.3.1.1 Finite analysis

The singular points  $(x_c, z_c)$  of the autonomous system (5.44) are obtained by setting the left hand sides of the equations to zero. They are displayed in table 5.1, where we also present the necessary conditions for their existence. In addition, for each singular point we calculate the values of  $w_M$  (given by relation (5.38)), of  $\Omega_{DE}$  (given by (5.37)), and of  $w_{DE}$  (given by (5.39)). Note that in this case,  $w_{DE}$  remains unspecified and the results hold independently of its value. The cosmological solutions associated with the singular points can be derived from the Raychaudhuri equation (5.42). Concerning the type and stability of singular points, for hyperbolic singular points (all the eigenvalues of the linearization matrix have real parts different from zero) one can easily extract their type (source (unstable) for positive real parts, saddle for real parts of different sign and sink (stable) for negative real parts). However, if at least one eigenvalue has a zero real part (non-hyperbolic singular point) one is not able to obtain conclusive information about the stability from linearization and needs to resort to other tools like Normal Forms calculations [359, 358], or numerical experimentation.

In the following we will discuss about the eigensystems (eigenvalues and associated eigenvectors) of the linearization evaluated at each singular points displayed in table 5.1. We summarize in table 5.1 their type and stability, acquired by examining the sign of the real part of the eigenvalues and some basic observables ( $w_M$ ,  $\Omega_{DE}$ , and  $w_{DE}$ ) evaluated at the singular points. In table 5.2 we display the corresponding cosmological solutions.

The singular points  $P_{1,2}$  exist for all the  $s$ -values. The singular points  $P_{1,2}$  have eigenvalues  $\{\mp 2\sqrt{6}s + 6, 1\}$  with associated eigenvectors  $\{1, 0\}$ ,  $\{0, 1\}$ . Thus, the singular points are of saddle type with a 1-dimensional unstable manifold tangent to the  $z$ -axis provided  $\pm s > \sqrt{\frac{3}{2}}$  (the sign  $+$  for  $P_1$  and the sign  $-$  for  $P_2$ ). Otherwise they are local sources.

The singular point  $P_3$  exists for  $s^2 < \frac{3}{2}$ . The singular point  $P_3$  has eigenvalues  $\{-3 +$

Table 5.1: Finite singular points of a flat universe with  $\Lambda = 0$  (case 1) and their behavior. NH stands for nonhyperbolic (adapted from [151]).

Cr. P	$x_c$	$z_c$	Existence	Stable for	$w_M$	$\Omega_{DE}$	$w_{DE}$
$P_{1,2}$	$\pm 1$	0	All $s$	unstable	1	0	arbitrary
$P_3$	$\sqrt{\frac{2}{3}}s$	0	$s^2 < \frac{3}{2}$	$s^2 < 1$	$\frac{4}{3}s^2 - 1$	0	arbitrary
$P_4$	$\sqrt{\frac{2}{3}}s$	$z_c$	$s = \pm 1$	NH	1/3	0	arbitrary

Table 5.2: Finite singular points and their corresponding solutions for a flat universe with  $\Lambda = 0$  (case 1).

Cr. P	Solution	Energy Density
$P_{1,2}$	$a \propto (t - t_0)^{\frac{1}{3}}$	$\rho_M \propto (t - t_0)^{-2}$
$P_3$	$a \propto (t - t_0)^{\frac{1}{2q^2}}$	$\rho_M \propto (t - t_0)^{-2}$
$P_4$	$a \propto (t - t_0)^{\frac{1}{2}}$	$\rho_M \propto (t - t_0)^{-2}$

$2s^2, 2(-1 + s^2)\}$  with associated eigenvectors  $\{1, 0\}, \{0, 1\}$ . Thus, the singular point is non-hyperbolic for  $s = \pm 1$ ; a sink provided  $-1 < s < 1$ . Otherwise, it is a saddle with 1-dimensional unstable manifold tangent to the  $z$ -axis.

Finally, note that in the special case where  $s = \pm 1$ , the system admits an extra curve of singular points  $P_4$ . Each point in  $P_4$  is non-hyperbolic, with center manifold tangent to the  $z$ -axis, but the curve  $P_4$  is actually “normally hyperbolic” [368]. This means that we can indeed analyze the stability by analyzing the sign of the real parts of the non-null eigenvalues. Therefore, since the non zero eigenvalue is negative,  $P_4$  is a local attractor.

In order to present the aforementioned behavior more transparently, we evolve the autonomous system (5.44) numerically for the choice  $s = 0.6$ , and the results are shown in figure 5.1. As we can see, in this case the singular point  $P_3$  is the global attractor of the system.

### 5.3.1.2 Analysis at infinity

Owing to the fact that the dynamical system (5.43)-(5.44) is non-compact, there could be features in the asymptotic regime which are non-trivial for the global dynamics. Thus, in order to complete the analysis of the phase space we will now extend our study using the Poincaré central projection method.

Let us introduce the Poincaré variables

$$x_r = \rho \cos \theta, \quad z_r = \rho \sin \theta, \quad (5.46)$$

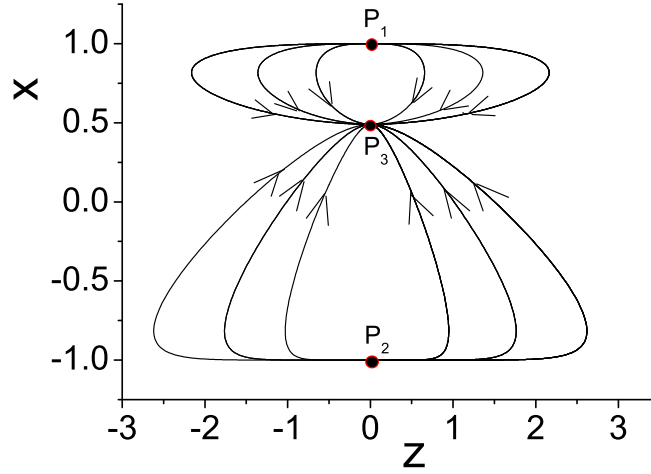


Figure 5.1: Phase plane for a flat universe with  $\Lambda = 0$  (case 1), for the choice  $s = 0.6$ . The singular points  $P_1$  and  $P_2$  are unstable (sources), while  $P_3$  is a global attractor. [Taken from [151] and published with permission of IOP Publishing Ltd]. See the global phase space in figure 5.2.

where  $\rho = \frac{r}{\sqrt{1+r^2}}$ ,  $r = \sqrt{x^2 + z^2}$  and  $\theta \in [0, 2\pi]$ . Thus, the points at “infinite” ( $r \rightarrow +\infty$ ) are those having  $\rho \rightarrow 1$ . The region of physical interest is given by

$$2x_r^2 + z_r^2 \leq 1.$$

Performing the transformation (5.46), the system (5.43)-(5.44) becomes

$$\begin{aligned} \rho' = & -s\rho^2\sqrt{6-6\rho^2}\cos^3\theta + \sqrt{6}s(1-\rho^2)^{3/2}\cos\theta + \\ & + \frac{1}{2}\rho(8\rho^2 + (4\rho^2 - 1)\cos(2\theta) - 5), \end{aligned} \quad (5.47)$$

$$\theta' = \left( \frac{\sqrt{6}s\rho\cos^2(\theta)}{\sqrt{1-\rho^2}} + \cos\theta - \frac{s\sqrt{6-6\rho^2}}{\rho} \right) \sin\theta. \quad (5.48)$$

In the limit  $\rho \rightarrow 1$ , the leading terms in (5.47) and in (5.48) are

$$\rho' \rightarrow 3\cos^2\theta, \quad (5.49)$$

$$\theta' \rightarrow \frac{\sqrt{6}s\cos^2(\theta)\sin\theta}{\sqrt{1-\rho^2}}. \quad (5.50)$$

Note that the radial equation does not contain the radial coordinate, so the singular points can be obtained using the angular equation only. Setting  $\theta' = 0$ , we obtain the singular points which are listed in table 5.3. The stability of these points is studied by analyzing first the stability of the angular coordinate and then deducing, from the sign of equation (5.47), the stability on the radial direction. We only need to perturb the angular variable  $\theta$  around the equilibrium points  $\theta_i$  via  $\theta = \theta_i + \delta\theta$ . The equilibrium points will be stable if  $\rho' > 0$  and the eigenvalue  $\lambda < 0$  for the linearized equation  $\delta\theta' = \lambda\delta\theta$ , in the

limit of  $\rho \rightarrow 1$ . When both conditions are satisfied the point is an stable node, if only one is satisfied it is a saddle and when neither holds it is an unstable node.

In table 5.3 are displayed the location of the asymptotic singular points of the system (5.43)-(5.44) (case 1) and their stability. The point  $Q_1$  is sink provided  $s > 0$  and a saddle otherwise. However this solution is unphysical since

$$2x_r^2 + z_r^2 > 1.$$

The singular points  $Q_2$  and  $Q_3$  are nonhyperbolic. Thus, we cannot anticipate its dynamical character from linearization. In such a case we can rely on numerical experimentation.

Table 5.3: Asymptotic singular points of the system (5.43)-(5.44) (case 1) and their stability.

Cr. P	Coordinates: $\theta, x_r, z_r$	Eigenvalue	$\rho'$	Stability
$Q_1$	$0, 1, 0$	$\begin{cases} -\infty & \text{for } s < 0 \\ +\infty & \text{for } s > 0 \end{cases}$	3	$\begin{cases} \text{sink} \\ \text{saddle} \end{cases}$
$Q_2$	$\frac{\pi}{2}, 0, 1$	0	0	nonhyperbolic
$Q_3$	$\frac{3\pi}{2}, 0, -1$	0	0	nonhyperbolic

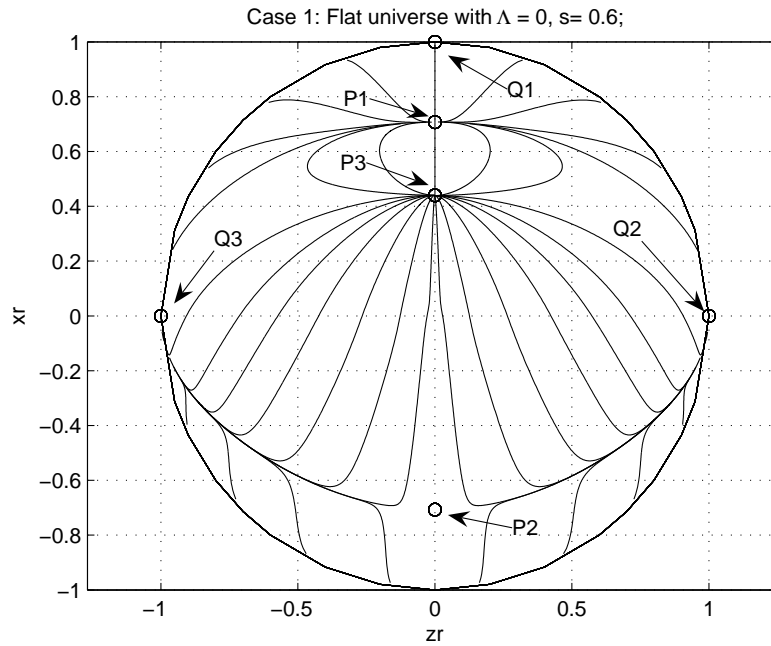


Figure 5.2: Poincaré projection (global phase space) of the system (5.43)-(5.44) (case 1) for the choice  $s = 0.6$ . Observe that the points at infinity  $Q_{1,2,3}$  are saddles, whereas the finite point  $P_3$  is the global attractor.

In figures 5.2 and 5.3 are drawn orbits in the global phase space of the system (5.43)-(5.44) (case 1). That is, the projection of the Poincaré sphere in the plane passing by its

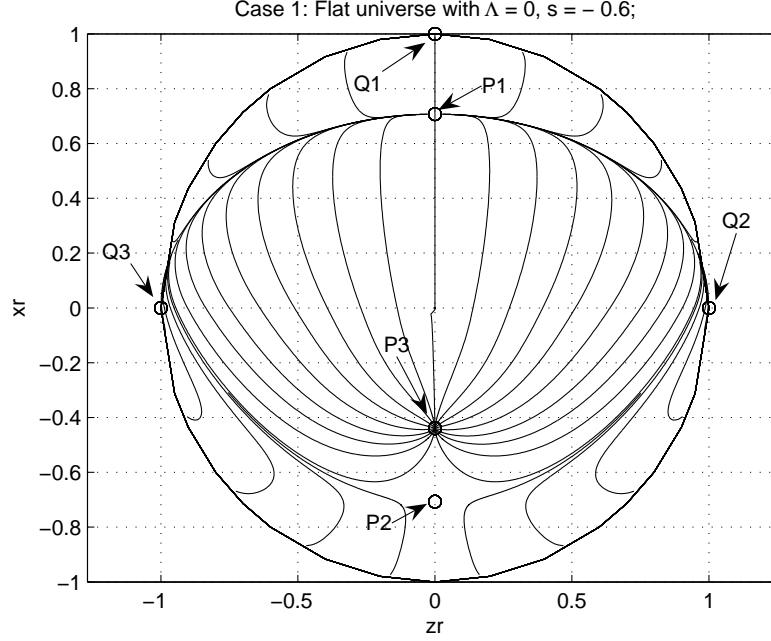


Figure 5.3: Global phase space of the system (5.43)-(5.44) (case 1) for the choice  $s = -0.6$ . The points at infinity  $Q_2$  and  $Q_3$  are saddles. The point  $Q_1$  is a local attractor at infinity (but it is unphysical) and  $P_3$  is a local attractor at the finite region.

equator. The points at infinity  $Q_i, i = 1, 2, 3$  are projected on the circle  $x_r^2 + z_r^2 = 1$ . For  $s > 0$ , the points at infinity  $Q_{1,2,3}$  are saddles, whereas the finite point  $P_3$  is the global attractor (see figure 5.2 for the choice  $s = 0.6$ ). For  $s < 0$  The points at infinity  $Q_2$  and  $Q_3$  are saddles. The point  $Q_1$  is a local attractor at infinity and  $P_3$  is a local attractor at the finite region (see figure 5.3 for the choice  $s = -0.6$ ).

### 5.3.1.3 Cosmological implications for Case 1: flat universe with $\Lambda = 0$

In this scenario the singular points  $P_{1,2}$  are not relevant from a cosmological point of view, since apart from being unstable they correspond to complete dark matter domination, with the matter equation-of-state parameter being unphysically stiff. However, point  $P_3$  is more interesting since it is stable for  $-1 < q < 1$  and thus it can be the late-time state of the universe. If additionally we desire to keep the dark-matter equation-of-state parameter in the physical range  $0 < w_M < 1$  then we have to restrict the parameter  $q$  in the range  $\sqrt{3}/2 < q < \sqrt{3}/2$ . However, even in this case the universe is finally completely dominated by dark matter. The fact that  $z_c = 0$  means that in general this sub-class of universes will be expand forever. The singular points  $P_4$  consist a stable late-time solution, with a physical dark-matter equation-of-state parameter  $w_M = 1/3$ , but with zero dark energy density. We mention that the dark-matter domination of the case at hand was expected, since in the absent of curvature and of a cosmological constant the corresponding Hořava-Lifshitz universe is comprised only by dark matter. Note however

that the dark-energy equation-of-state parameter can be arbitrary. We have obtained an unphysical attractor at the infinity region.

### 5.3.2 Case 2: non-flat universe with $\Lambda = 0$

Under this scenario, and using the auxiliary variables (5.32),(5.33),(5.34),(5.35), the Friedmann equations (5.17), (5.18) become:

$$1 = x^2 + y^2 - z^2 \quad (5.51)$$

$$\frac{H'}{H} = -3x^2 + 2z^2, \quad (5.52)$$

while the autonomous system writes:

$$x' = x(3x^2 - 2z^2 - 3) + \sqrt{6}s(-x^2 + z^2 + 1), \quad (5.53)$$

$$z' = z[3x^2 - 2(z^2 + 1)]. \quad (5.54)$$

It is defined in the phase space  $\Psi = \{(x, z) : x^2 - z^2 \leq 1, z \in \mathbb{R}\}$  and as before the phase space is not compact.

#### 5.3.2.1 Finite analysis

The singular points, the conditions for their existence and stability, and the physical quantities are presented in table 5.4. Thus,  $P_{1,2,3}$  are exactly the same as in case 1, while  $P_{5,6}$  are saddle points except if  $s^2 \rightarrow 1$ , where they are nonhyperbolic because they give rise to the eigenvalues  $\{-\frac{1}{2} - \frac{1}{2}\mu_0, -\frac{1}{2} + \frac{1}{2}\mu_0\}$  with associated eigenvectors  $\{\pm\mu_1, 1\}, \{\pm\mu_2, 1\}$ , where we use the notations  $\mu_0 = \sqrt{-15 + \frac{16}{s^2}}$ ,  $\mu_1 = \frac{-9s^2 - \sqrt{16s^2 - 15s^4 + 8}}{4\sqrt{\frac{6}{s^2} - 6s}}$  and  $\mu_2 = \frac{-9s^2 + \sqrt{16s^2 - 15s^4 + 8}}{4\sqrt{\frac{6}{s^2} - 6s}}$ . It is interesting to notice that this scenario admits two more unstable singular points, namely  $P_{7,8}$ , in which  $z_c^2 = -1$ . The eigenvalues of the linearization are  $\{4, -1\}$  with associated eigenvectors  $\{\pm\frac{2}{5}i\sqrt{6}s, 1\}, \{1, 0\}$ . These points are of great physical importance, as we are going to see in the next section.

In order to present the results more transparently, in fig. 5.4 we present the numerical evolution of the system for the choice  $s = \sqrt{3}$ . In this specific realization of the scenario the singular points  $P_3$  and  $P_{5,6,7,8}$  do not exist. We find only the source  $P_1$  and the saddle  $P_2$ , and we indeed observe that there is one orbit approaching  $P_2$  (the solution with  $z \equiv 0$ ). Finally, note that the divergence of the orbits towards the future is typical and suggests that the future attractor of the system can be located at infinite regions.

In fig. 5.5 we depict the phase-space graph for the choice  $s = 0.6$ . In this case the singular points  $P_{1,2}$  are unstable (sources), while  $P_3$  is a local attractor. The points  $P_{5,6}$  are saddle ones, and thus we observe that some orbits coming from infinity spend a large amount of time near them before diverge again in a finite time.

Table 5.4: The singular points of a non-flat universe with  $\Lambda = 0$  (case 2) and their behavior (adapted from [151]).

Cr. P	$x_c$	$z_c$	Existence	Stable for	$w_M$	$\Omega_{DE}$	$w_{DE}$
$P_{1,2}$	$\pm 1$	0	All $s$	unstable	1	0	arbitrary
$P_3$	$\sqrt{\frac{2}{3}}s$	0	$s^2 < \frac{3}{2}$	$s^2 < 1$	$\frac{4}{3}s^2 - 1$	0	arbitrary
$P_{5,6}$	$\sqrt{\frac{3}{2}}\frac{1}{s}$	$\pm\sqrt{-1 + \frac{1}{s^2}}$	$s^2 \leq 1, s \neq 0$	unstable	$\frac{3}{s^2} - 1$	0	arbitrary
$P_{7,8}$	0	$\pm i$	always	unstable	arbitrary	1	1/3

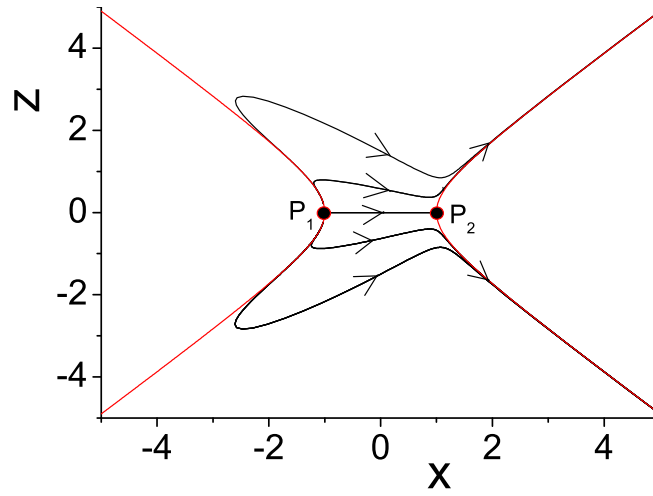


Figure 5.4: Phase plane for a non-flat universe with  $\Lambda = 0$  (case 2), for the choice  $s = \sqrt{3}$ . In this specific scenario the singular points  $P_3$  and  $P_{5,6,7,8}$  do not exist, while  $P_1$  and  $P_2$  are unstable (source and saddle respectively). [Taken from [151] and published with permission of IOP Publishing Ltd]. See the global phase space in figure 5.6.

### 5.3.2.2 Analysis at infinity

Using the same procedure as in section 5.3.1.2; that is, performing the transformation (5.46), the system (5.53)-(5.54) becomes

$$\rho' = 3\rho^3 - \frac{1}{2}s\sqrt{6-6\rho^2}\cos(3\theta)\rho^2 - \frac{5\rho}{2} + \frac{1}{2}s\sqrt{6-6\rho^2}(2-3\rho^2)\cos\theta + \left(3\rho^3 - \frac{\rho}{2}\right)\cos(2\theta), \quad (5.55)$$

$$\theta' = \cos\theta\sin(\theta) + \frac{\sqrt{6}s\rho\cos(2\theta)\sin\theta}{\sqrt{1-\rho^2}} - \frac{s\sqrt{6-6\rho^2}\sin\theta}{\rho}. \quad (5.56)$$



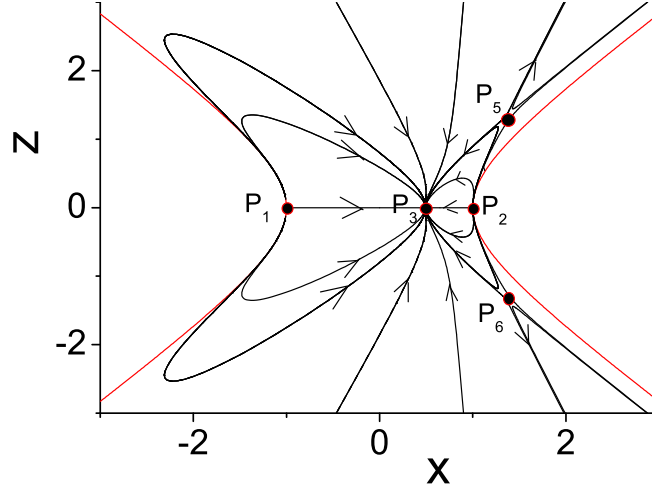


Figure 5.5: Phase plane for a non-flat universe with  $\Lambda = 0$  (case 2), for the choice  $s = 0.6$ . In this specific scenario the singular point  $P_3$  is a local attractor, while  $P_{1,2}$  are unstable (sources) and  $P_{5,6}$  are saddle ones. [Taken from [151] and published with permission of IOP Publishing Ltd]. See the global phase space in figure 5.7.

In this case the physical region is given by

$$-\frac{\sqrt{2}}{2} \leq x_r \leq \frac{\sqrt{2}}{2}, \quad x_r^2 + z_r^2 \leq 1.$$

In the limit  $\rho \rightarrow 1$ , the leading terms in (5.47) and in (5.48) are

$$\rho' \rightarrow \frac{1}{2}(5 \cos(2\theta) + 1), \quad (5.57)$$

$$\theta' \rightarrow \frac{\sqrt{6}s \cos(2\theta) \sin \theta}{\sqrt{1 - \rho^2}}. \quad (5.58)$$

As before, the radial equation does not contain the radial coordinate, so the singular points can be obtained using the angular equation only. Setting  $\theta' = 0$ , we obtain the singular points which are listed in table 5.5. The stability of these points is studied by analyzing first the stability of the angular coordinate and then deducing, from the sign of equation (5.55), the stability on the radial direction. In table 5.5 are displayed the asymptotic singular points of the system (5.53)-(5.54) (case 2). We comment there on their stability. For  $s < 0$  the local sinks are  $Q_4$  and  $Q_6$ ; the rest of the points at infinity are saddles. For  $s > 0$  the local sinks are  $Q_{5,7,8}$  whereas the rest of the points at infinity are saddles.

The system (5.55)-(5.56) have an apparent singularity at  $\rho = 0, \sin \theta = 0$  which is due to the spherical coordinate system. Thus, for numerical integrations is more convenient

Table 5.5: Asymptotic singular points of the system (5.53)-(5.54) (case 2) and their stability.

Cr. P	Coordinates: $\theta, x_r, z_r$	Eigenvalue	$\rho'$	Stability
$Q_4$	$0, 1, 0$	$\begin{cases} -\infty & \text{for } s < 0 \\ +\infty & \text{for } s > 0 \end{cases}$	3	$\begin{cases} \text{sink} \\ \text{saddle} \end{cases}$
$Q_5$	$\frac{\pi}{4}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$	$\begin{cases} +\infty & \text{for } s < 0 \\ -\infty & \text{for } s > 0 \end{cases}$	$\frac{1}{2}$	$\begin{cases} \text{saddle} \\ \text{sink} \end{cases}$
$Q_6$	$\frac{3\pi}{4}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$	$\begin{cases} -\infty & \text{for } s < 0 \\ +\infty & \text{for } s > 0 \end{cases}$	$\frac{1}{2}$	$\begin{cases} \text{sink} \\ \text{saddle} \end{cases}$
$Q_7$	$\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$	$\begin{cases} +\infty & \text{for } s < 0 \\ -\infty & \text{for } s > 0 \end{cases}$	$\frac{1}{2}$	$\begin{cases} \text{saddle} \\ \text{sink} \end{cases}$
$Q_8$	$\frac{7\pi}{4}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$	$\begin{cases} +\infty & \text{for } s < 0 \\ -\infty & \text{for } s > 0 \end{cases}$	$\frac{1}{2}$	$\begin{cases} \text{saddle} \\ \text{sink} \end{cases}$

to use the cartesian coordinates  $x_r, z_r$ . The system reads

$$\begin{aligned} x'_r &= 6x_r^3 - 3x_r + \frac{\sqrt{6}s(2x_r^4 - 3x_r^2 + 1)}{\sqrt{1 - x_r^2 - z_r^2}}, \\ z'_r &= (6x_r^2 - 2)z_r + \frac{\sqrt{6}s x_r(2x_r^2 - 1)z_r}{\sqrt{1 - x_r^2 - z_r^2}}. \end{aligned} \quad (5.59)$$

To illustrate the global dynamics we depict in the figure 5.6 the phase space of the system (5.59) describing the flow of (5.53)-(5.54) (case 2) for the choice  $s = \sqrt{3}$  in the Poincaré variables. As in figure 5.4 the singular points  $P_3$  and  $P_{5,6,7,8}$  do not exist. At finite region we find only the source  $P_1$  and the saddle  $P_2$ , and we indeed observe that there is one orbit approaching  $P_2$  (the solution with  $z \equiv 0$ ). The future attractors of the system are  $Q_5$  and  $Q_8$  located at the infinite region. The points at infinity  $Q_{4,6,7}$  are saddles.  $Q_4$  is unphysical since  $x \notin \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ . Also, in the figure 5.7 we depict the global phase space for case 2 for the choice  $s = 0.6$ . The finite points  $P_1$  and  $P_2$  are local sources.  $P_3$  is a local sink at the finite region. This is consistent with the drawn in figure 5.5. Observe that the orbits spent a finite lapse of time near the saddle points  $P_5$  and  $P_6$  before reaching the local sinks at infinity  $Q_5$  and  $Q_8$  respectively. The points at infinity  $Q_{4,6,7}$  are saddles. As in the previous figure,  $Q_4$  is unphysical.

### 5.3.2.3 Cosmological implications for Case 2: non-flat universe with $\Lambda = 0$

In this scenario, the first three singular points are identical with those of case 1, and thus the physical implications are the same. The singular points  $P_{5,6}$  are unstable, corresponding to a dark-matter dominated universe. This was expected since in the absence of the

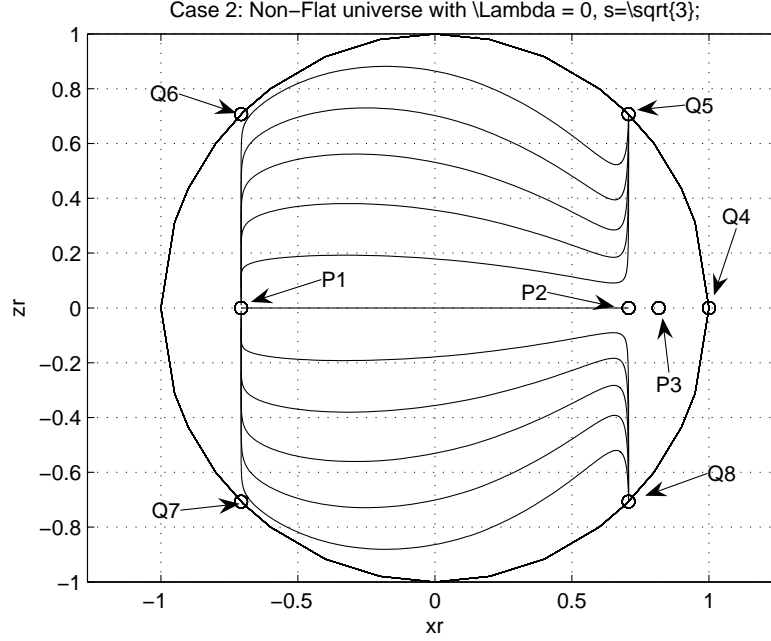


Figure 5.6: Global phase space of the system (5.53)-(5.54) (case 2) for the choice  $s = \sqrt{3}$ . The singular points  $P_3$  and  $P_{5,6,7,8}$  do not exist. At finite region we find only the source  $P_1$  and the saddle  $P_2$ , and we indeed observe that there is one orbit approaching  $P_2$  (the solution with  $z \equiv 0$ ). The future attractors of the system are  $Q_5$  and  $Q_8$  located at the infinite region. The points at infinity  $Q_{4,6,7}$  are saddles.  $Q_4$  is unphysical since  $x \notin \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ .

cosmological constant  $\Lambda$ , the curvature role is downgrading as the scale factor increases and thus in the end this case tends to the case 1 above. Note however that at early times, where the scale factor is small, the behavior of the system will be significantly different than case 1, with the dark energy playing an important role. This different behavior is observed in the corresponding phase-space figures 5.4, 5.5 comparing with figure 5.1.

The case at hand admits another solution sub-class, namely points  $P_{7,8}$ . In these points  $z_c^2 = -1$ , and thus using (5.34) we straightforwardly find the late-time solution  $a(t) = e^{i\gamma t}$ , with  $\gamma = |\kappa^2 \mu / [4(3\lambda - 1)]|$ . This solution corresponds to an oscillatory universe [439, 440], and in the context of Hořava-Lifshitz cosmology it has already been studied in the literature [282, 283, 284]. However, as we see, these singular points are unstable and thus this solution subclass cannot be a late-time attractor in the case of a non-flat universe with zero cosmological constant. This situation will change in the case where the cosmological constant is switched on.

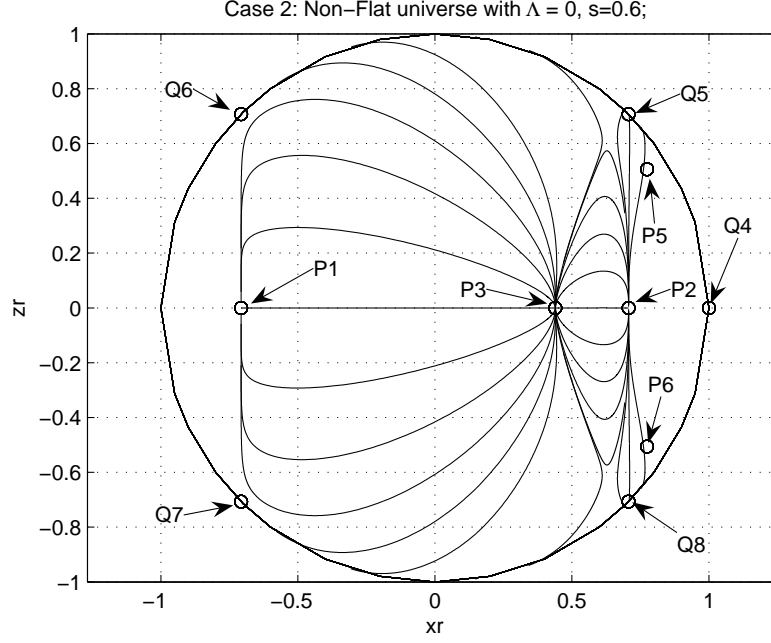


Figure 5.7: Global phase space of the system (5.53)-(5.54) (case 2) for the choice  $s = 0.6$ . The finite points  $P_1$  and  $P_2$  are local sources.  $P_3$  is a local sink at the finite region. Observe that the orbits spent a finite lapse of time near the saddle points  $P_5$  and  $P_6$  before reaching the local sinks at infinity  $Q_5$  and  $Q_8$  respectively. The points at infinity  $Q_{4,6,7}$  are saddles.  $Q_4$  is unphysical since  $x \notin \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ .

### 5.3.3 Case 3: flat universe with $\Lambda \neq 0$

In this case the Friedmann equations (5.17), (5.18) write as

$$1 = x^2 + y^2 - u^2 \quad (5.60)$$

$$\frac{H'}{H} = -3x^2, \quad (5.61)$$

and the autonomous system becomes:

$$x' = \sqrt{6}s(u^2 - x^2 + 1) + 3x(x^2 - 1), \quad (5.62)$$

$$u' = 3ux^2, \quad (5.63)$$

defined in the phase space  $\Psi = \{(x, u) : x^2 - u^2 \leq 1, u \in \mathbb{R}\}$ . As before the phase space is not compact.

#### 5.3.3.1 Finite analysis

The singular points, the conditions for their existence and stability, and the physical quantities are presented in table 5.6.

The singular points  $P_{9,10}$  exist for all the  $s$ -values. They have eigenvalues  $\mp 2\sqrt{6}s +$



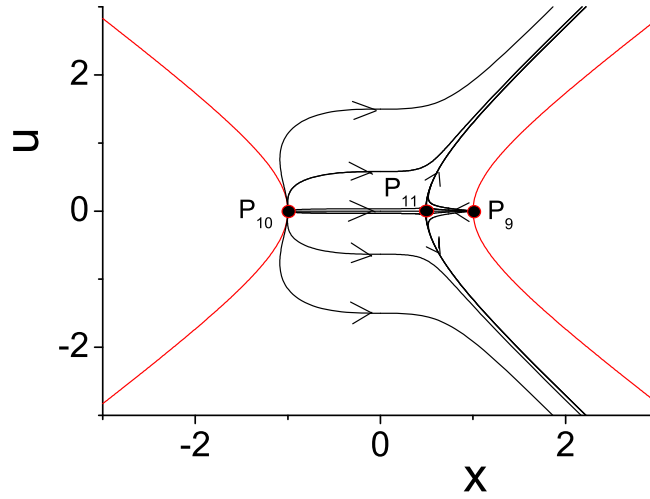


Figure 5.9: Phase plane for a flat universe with  $\Lambda \neq 0$  (case 3), for the choice  $s = 0.6$ . In this specific scenario the singular point  $P_{11}$  is a saddle one, while  $P_9$  and  $P_{10}$  are unstable (sources). [Taken from [151] and published with permission of IOP Publishing Ltd]. See the global phase space in figure 5.11.

singular point  $P_{11}$  is a saddle one (with stable manifold tangent to the  $x$ -axis), while  $P_9$  and  $P_{10}$  are unstable (sources). There are two orbits, one joining  $P_{10}$  with  $P_{11}$  and one joining  $P_9$  with  $P_{11}$ , both of them overlapping the  $x$ -axis. Note that some orbits remain close to  $P_{11}$  before finally diverge towards the future, and this suggests that the future attractor of the system is located at infinite regions.

### 5.3.3.2 Analysis at infinity

Table 5.7: Asymptotic singular points of the system (5.62)-(5.63) (case 3) and their stability.

Cr. P	Coordinates: $\theta, x_r, u_r$	Eigenvalue	$\rho'$	Stability
$Q_9$	$0, 1, 0$	$\begin{cases} -\infty & \text{for } s < 0 \\ +\infty & \text{for } s > 0 \end{cases}$	3	$\begin{cases} \text{sink} \\ \text{saddle} \end{cases}$
$Q_{10}$	$\frac{\pi}{4}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$	$\begin{cases} +\infty & \text{for } s < 0 \\ -\infty & \text{for } s > 0 \end{cases}$	$\frac{3}{2}$	$\begin{cases} \text{saddle} \\ \text{sink} \end{cases}$
$Q_{11}$	$\frac{3\pi}{4}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$	$\begin{cases} -\infty & \text{for } s < 0 \\ +\infty & \text{for } s > 0 \end{cases}$	$\frac{3}{2}$	$\begin{cases} \text{sink} \\ \text{saddle} \end{cases}$
$Q_{12}$	$\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$	$\begin{cases} -\infty & \text{for } s < 0 \\ +\infty & \text{for } s > 0 \end{cases}$	$\frac{3}{2}$	$\begin{cases} \text{sink} \\ \text{saddle} \end{cases}$
$Q_{13}$	$\frac{7\pi}{4}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$	$\begin{cases} +\infty & \text{for } s < 0 \\ -\infty & \text{for } s > 0 \end{cases}$	$\frac{3}{2}$	$\begin{cases} \text{saddle} \\ \text{sink} \end{cases}$

The coordinate transformation

$$x_r = \rho \cos \theta, \quad u_r = \rho \sin \theta, \quad (5.64)$$

where  $\rho = \frac{r}{\sqrt{1+r^2}}$ , and  $r = \sqrt{x^2 + u^2}$ , allows to investigate the asymptotics of the system (5.62)-(5.63) (i.e., at the region  $r \rightarrow +\infty$ ) by taking the limit  $\rho \rightarrow 1$ . In this case the physical region is given by

$$-\frac{\sqrt{2}}{2} \leq x_r \leq \frac{\sqrt{2}}{2}, \quad x_r^2 + u_r^2 \leq 1.$$

Performing the transformation (5.64), the system (5.62)-(5.63) becomes

$$\rho' = (6\rho^3 - 3\rho) \cos^2 \theta + \left( \sqrt{6}s (1 - \rho^2)^{3/2} - s\rho^2 \sqrt{6 - 6\rho^2} \cos(2\theta) \right) \cos \theta, \quad (5.65)$$

$$\theta' = 3 \cos \theta \sin \theta + \frac{\sqrt{6}s \rho \cos(2\theta) \sin \theta}{\sqrt{1 - \rho^2}} - \frac{s \sqrt{6 - 6\rho^2} \sin \theta}{\rho}. \quad (5.66)$$

In the limit  $\rho \rightarrow 1$ , the leading terms in (5.65)-(5.66) are

$$\rho' \rightarrow 3 \cos^2 \theta, \quad (5.67)$$

$$\theta' = \frac{\sqrt{6}s \cos(2\theta) \sin \theta}{\sqrt{1 - \rho^2}}. \quad (5.68)$$

As before, the radial equation does not contain the radial coordinate, so the singular points can be obtained using the angular equation only. Setting  $\theta' = 0$ , we obtain the singular points which are listed in table 5.7. The stability of these points is studied by analyzing first the stability of the angular coordinate and then deducing, from the sign of equation (5.65), the stability on the radial direction.

The system (5.65)-(5.66) have an apparent singularity at  $\rho = 0, \sin \theta = 0$  which is due to the spherical coordinate system. Thus, for numerical integrations is more convenient to use the cartesian coordinates  $x_r, u_r$ . The system reads

$$\begin{aligned} x_r' &= 6x_r^3 - 3x_r + \frac{\sqrt{6}s (2x_r^4 - 3x_r^2 + 1)}{\sqrt{1 - u_r^2 - x_r^2}}, \\ u_r' &= 6u_r x_r^2 + \frac{\sqrt{6}s u_r (2x_r^2 - 1) x_r}{\sqrt{1 - u_r^2 - x_r^2}}. \end{aligned} \quad (5.69)$$

To illustrate the global dynamics we depicts in the figure 5.10 the phase space of the system (5.69) describing the flow of (5.62)-(5.63) (case 3) for the choice  $s = \sqrt{3}$  in the Poincaré variables. As in figure 5.8, the singular point  $P_{11}$  does not exists.  $P_{10}$  is unstable (source), while  $P_9$  is a saddle one. The orbits passing near the saddle  $P_9$  bifurcates and tends asymptotically to one of the global attractor at infinity  $Q_{10}$  or  $Q_{13}$  depending on the

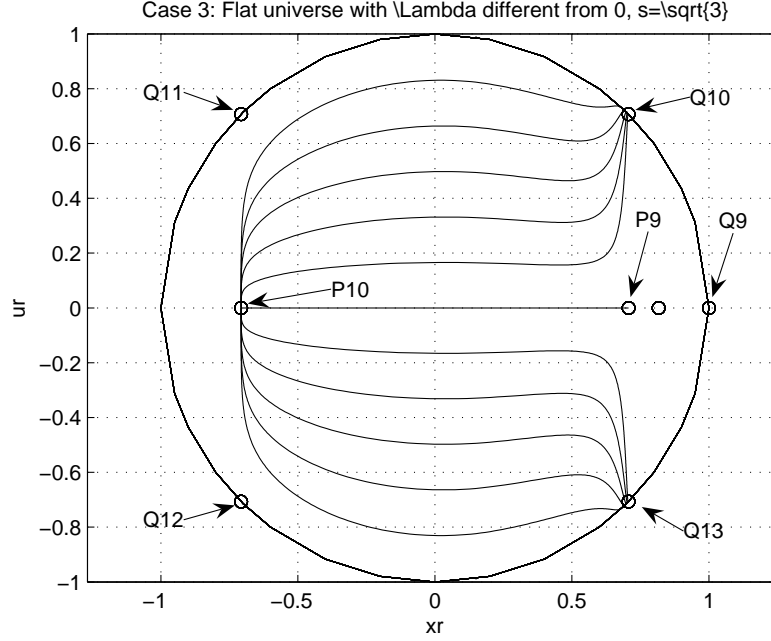


Figure 5.10: Global phase space of the system (5.43)-(5.44) (case 3) for the choice  $s = \sqrt{3}$ . As in figure 5.8, the singular point  $P_{11}$  does not exist.  $P_{10}$  is unstable (source), while  $P_9$  is a saddle one. The orbits passing near the saddle  $P_9$  bifurcates and tends asymptotically to one of the global attractor at infinity  $Q_{10}$  or  $Q_{13}$  depending on the sign of the initial value of  $u_r$ . The points at infinity  $Q_{9,11,12}$  are saddles;  $Q_9$  is unphysical.

sign of the initial value of  $u_r$ . The points at infinity  $Q_{9,11,12}$  are saddles;  $Q_9$  is unphysical. In figure (5.11) we drawn the global phase space for the choice  $s = 0.6$ . As in figure 5.9, in this specific scenario the singular point  $P_{11}$  is a saddle one, while  $P_9$  and  $P_{10}$  are unstable (sources). The global attractors at infinity are the points  $Q_{10}$  and  $Q_{13}$ . The points at infinity  $Q_{9,11,12}$  are saddles;  $Q_9$  is unphysical.

### 5.3.3.3 Cosmological implications for Case 3: flat universe with $\Lambda \neq 0$

Under this scenario, the Hořava-Lifshitz universe admits two unstable singular points ( $P_{9,10}$ ), completely dominated by stiff dark matter. Point  $P_{11}$  exhibits a more physical dark matter equation-of-state parameter, but still with negligible dark energy at late times. The case at hand admits the two nonhyperbolic points  $P_{12,13}$  possessing  $u_c^2 = -1$ , and thus (as can be seen by (5.35)) they correspond to the oscillatory solution  $a(t) = e^{i\delta t}$ , with  $\delta = |\kappa^2 \mu \Lambda / [4(3\lambda - 1)]|$ . We mention that these points are nonhyperbolic, with a negative eigenvalue, and thus they have a large probability to be a late-time solution of Hořava-Lifshitz universe. Additionally, they correspond to dark-energy domination, with dark-energy equation-of-state parameter  $-1$  and an arbitrary  $w_M$ . These features make them good candidates to be a realistic description of the universe. We mention that this result is independent from the parameter  $q$  which comes from the dark matter sector. Thus,



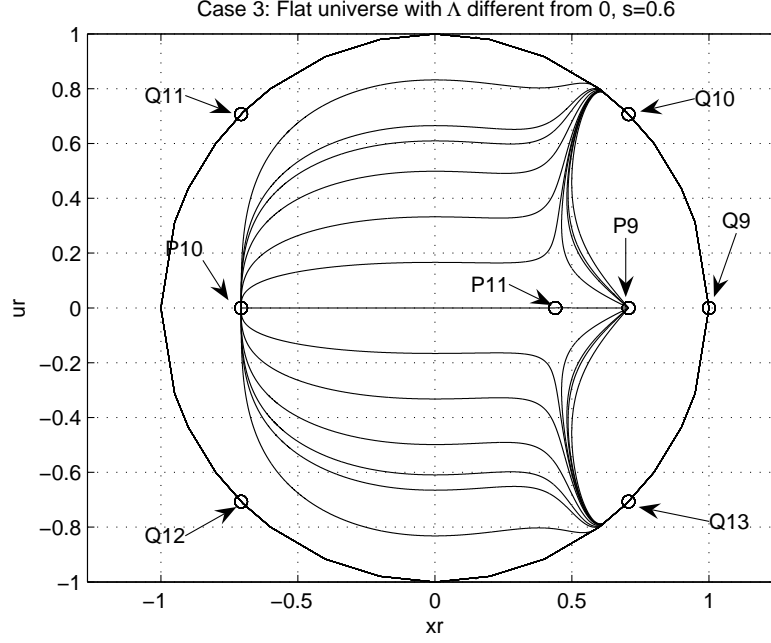


Figure 5.11: Global phase space of the system (5.62)-(5.63) (case 3) for the choice  $s = 0.6$ . As in figure 5.9, in this specific scenario the singular point  $P_{11}$  is a saddle one, while  $P_9$  and  $P_{10}$  are unstable (sources). The global attractors at infinity are the points  $Q_{10}$  and  $Q_{13}$ . The points at infinity  $Q_{9,11,12}$  are saddles;  $Q_9$  is unphysical.

we conclude that it is valid independently of the matter-content of the universe. Indeed, this behavior is novel, and arises purely by the extra terms that are present in Hořava gravity.

#### 5.3.4 Case 4: $k \neq 0, \Lambda \neq 0$

Under this scenario, and using the auxiliary variables (5.32),(5.33),(5.34),(5.35), the Friedmann equations (5.17), (5.18) become:

$$1 = x^2 + y^2 - (u - kz)^2 \quad (5.70)$$

$$\frac{H'}{H} = -3x^2 + 2z(-u + z). \quad (5.71)$$

while the autonomous system writes:

$$\begin{aligned} x' &= \sqrt{6}s [-x^2 + (u - z)^2 + 1] + x [3x^2 + 2(u - z)z - 3], \\ z' &= z [3x^2 + 2(u - z)z - 2], \\ u' &= u [3x^2 + 2(u - z)z], \end{aligned} \quad (5.72)$$

defined in the phase space  $\Psi = \{(x, z, u) : x^2 - (u - kz)^2 \leq 1, u, z \in \mathbb{R}\}$ , which is not compact.

Table 5.8: The singular points of a non-flat universe with  $\Lambda \neq 0$  (case 4) and their behavior (adapted from [151]).

Cr. P	$x_c$	$z_c$	$u_c$	Existence	Stable for	$w_M$	$\Omega_{DE}$	$w_{DE}$
$P_{14,15}$	$\pm 1$	0	0	All $s$	unstable	1	0	arbitrary
$P_{16}$	$\sqrt{\frac{2}{3}}s$	0	0	$s^2 < \frac{3}{2}$	unstable	$\frac{4}{3}s^2 - 1$	0	arbitrary
$P_{17,18}$	$\sqrt{\frac{3}{2}}\frac{1}{s}$	$\pm\sqrt{-1 + \frac{1}{s^2}}$	0	$s^2 \leq 1, s \neq 0$	unstable	2	$1 - \frac{1}{s^2}$	1/3
$P_{19,20}$	0	0	$\pm i$	always	NH	arbitrary	1	-1
$P_{21,22}$	0	$\pm i$	0	always	unstable	arbitrary	1	1/3

#### 5.3.4.1 Finite analysis

The singular points, and their corresponding information, are presented in table 5.8.

The singular point  $P_{14}$  is nonhyperbolic if  $s = \sqrt{3/2}$ , it is a source if  $s < \sqrt{3/2}$  or a saddle otherwise, while  $P_{15}$  is nonhyperbolic if  $s = -\sqrt{3/2}$ , it is a source if  $s > -\sqrt{3/2}$  or a saddle otherwise.  $P_{16}$  is nonhyperbolic if  $s^2 \in \{0, 1, 3/2\}$  and saddle otherwise, while  $P_{17,18}$  are nonhyperbolic if  $s^2 \rightarrow 1$ , and saddle otherwise. The points  $P_{19,20}$  have the eigenvalues  $\{-3, -2, 0\}$  with associated eigenvectors  $\{1, 0, 0\}, \{0, 1, 1\}, \{\pm 2i\sqrt{\frac{2}{3}}s, 0, 1\}$ . Hence, they are nonhyperbolic possessing a 2-dimensional stable manifold. Finally,  $P_{21,22}$  are unstable because they give rise to the eigenvalues  $\{4, 2, -1\}$ , with associated eigenvectors  $\{-\frac{2}{5}i\sqrt{6}s, 1, 0\}, \{0, 1, 1\}, \{1, 0, 0\}$ .

#### 5.3.4.2 Analysis at infinity

The coordinate transformation

$$x_r = \rho \cos \theta \sin \psi, \quad z_r = \rho \sin \theta \sin \psi, \quad u_r = \rho \cos \psi, \quad (5.73)$$

where  $\rho = \frac{r}{\sqrt{1+r^2}}$ , and  $r = \sqrt{x^2 + z^2 + u^2}$ ,  $\theta \in [0, 2\pi]$ ,  $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ , allows to investigate the asymptotics of the system (5.72)-(5.72) (i.e., at the region  $r \rightarrow +\infty$ ) by taking the limit  $\rho \rightarrow 1$ . In this case the physical region is given by

$$2(x_r^2 + ku_r z_r) \leq 1, \quad u_r^2 + x_r^2 + z_r^2 \leq 1.$$

Table 5.9: Asymptotic singular points of the system (5.72)-(5.72) (case 4) and their stability. NH stands for nonhyperbolic

Cr. P	Coordinates $\theta, \psi, x_r, x_r, u_r$	Eigenvalues	$\rho'$	Stability
$Q_{14}$	$0, \frac{\pi}{4}, \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}$	$\begin{cases} 0, -\infty & \text{for } s > 0 \\ 0, +\infty & \text{for } s < 0 \end{cases}$	$-\frac{3}{2}$	$\begin{cases} \text{NH, 1D stable manifold} \\ \text{NH, 2D unstable manifold} \end{cases}$
$Q_{15}$	$\frac{\pi}{2}, \frac{\pi}{4}, 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$	$0, 0$	$0$	NH
$Q_{16}$	$\pi, \frac{\pi}{4}, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}$	$\begin{cases} 0, +\infty & \text{for } s > 0 \\ 0, -\infty & \text{for } s < 0 \end{cases}$	$-\frac{3}{2}$	$\begin{cases} \text{NH, 2D unstable manifold} \\ \text{NH, 1D stable manifold} \end{cases}$
$Q_{17}$	$0, -\frac{\pi}{4}, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}$	$\begin{cases} 0, -\infty & \text{for } s > 0 \\ 0, +\infty & \text{for } s < 0 \end{cases}$	$-\frac{3}{2}$	$\begin{cases} \text{NH, 1D stable manifold} \\ \text{NH, 2D unstable manifold} \end{cases}$
$Q_{18}$	$\pi, -\frac{\pi}{4}, -\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}$	$\begin{cases} 0, +\infty & \text{for } s > 0 \\ 0, -\infty & \text{for } s < 0 \end{cases}$	$-\frac{3}{2}$	$\begin{cases} \text{NH, 2D unstable manifold} \\ \text{NH, 1D stable manifold} \end{cases}$
$Q_{19}$	$0, \frac{\pi}{2}, 1, 0, 0$	$\begin{cases} +\infty, +\infty & \text{for } s > 0 \\ -\infty, -\infty & \text{for } s < 0 \end{cases}$	$-3$	$\begin{cases} \text{source} \\ \text{saddle} \end{cases}$
$Q_{20}$	$0, -\frac{\pi}{2}, -1, 0, 0$	$\begin{cases} -\infty, -\infty & \text{for } s > 0 \\ +\infty, +\infty & \text{for } s < 0 \end{cases}$	$-3$	$\begin{cases} \text{saddle} \\ \text{source} \end{cases}$

Performing the transformation (5.73), the system (5.72)-(5.72) becomes, as  $\rho \rightarrow 1$

$$\rho' \rightarrow -\frac{1}{2} \sin \psi (4 \cos \psi \sin \theta + 5 \cos(2\theta) \sin \psi + \sin \psi), \quad (5.74)$$

$$\theta' \rightarrow \frac{\sqrt{6}s \sin \theta (2 \sin \psi \cos^2 \theta - \csc \psi + 2 \cos \psi \sin \theta)}{\sqrt{1 - \rho^2}}, \quad (5.75)$$

$$\psi' \rightarrow \frac{\sqrt{\frac{3}{2}}s \cos \psi (2 \cos(2\psi) \cos^3 \theta + 2 \sin^2 \theta \cos \theta - \sin(2\theta) \sin(2\psi))}{\sqrt{1 - \rho^2}}. \quad (5.76)$$

As before, the radial equation does not contain the radial coordinate, so the singular points can be obtained using the angular equations only. Setting  $\theta' = 0, \psi' = 0$ , we obtain the singular points which are listed in table 5.9. The stability of these points is studied by analyzing first the stability of the angular coordinates and then deducing, from the sign of equation (5.74), the stability on the radial direction.

### 5.3.4.3 Cosmological implications for Case 4: non-flat universe with $\Lambda \neq 0$

This case admits the unstable singular points  $P_{14,15,16}$  which correspond to a dark-matter dominated universe, and the unstable points  $P_{17,18}$  which are unphysical since they possess  $w_M = 2$ . As expected, the system admits also the unstable points  $P_{21,22}$  which correspond to oscillatory universes with  $a(t) = e^{i\gamma t}$  ( $\gamma = |\kappa^2 \mu / [4(3\lambda - 1)]|$ ). However, we find

two more oscillatory singular points, namely  $P_{19,20}$ , which correspond to  $a(t) = e^{i\delta t}$ , with  $\delta = |\kappa^2 \mu \Lambda / [4(3\lambda - 1)]|$ . These points are nonhyperbolic, with a negative eigenvalue, and thus they have a large probability to be the late-time state of the universe, and additionally this result is independent of the specific form of the dark-matter content. Furthermore, they correspond to a dark-energy dominated universe, with  $w_{DE} = -1$  and arbitrary  $w_M$ . Thus, they are good candidates for a realistic description of the universe.

## 5.4 Beyond detailed balance: phase space analysis

In this section we extend the phase-space analysis to a universe governed by Hořava gravity in which the detailed balance condition has been relaxed. In order to transform the corresponding cosmological equations into an autonomous dynamical system, we use the auxiliary variables  $x$  and  $y$  defined in (5.32),(5.33), and furthermore we define the following four new ones:

$$\begin{aligned} x_1 &= \frac{\sigma_1}{3(3\lambda - 1)H^2}, \\ x_2 &= \frac{k\sigma_2}{3(3\lambda - 1)a^2H^2}, \\ x_3 &= \frac{\sigma_3}{3(3\lambda - 1)a^4H^2}, \\ x_4 &= \frac{2k\sigma_4}{(3\lambda - 1)a^6H^2}. \end{aligned} \quad (5.77)$$

Thus, using these variables and the definitions (5.30) and (5.31), we can express the dark energy density and equation-of-state parameters respectively as:

$$\begin{aligned} \Omega_{DE}|_{\text{non-db}} &\equiv \frac{2}{(3\lambda - 1)H^2} \left( \frac{\sigma_1}{6} + \frac{\sigma_3 k^2}{6a^4} + \frac{\sigma_4 k}{a^6} \right) = \\ &= x_1 + x_3 + x_4, \end{aligned} \quad (5.78)$$

$$w_{DE}|_{\text{non-db}} \equiv \frac{-\frac{\sigma_1}{6} + \frac{\sigma_3 k^2}{18a^4} + \frac{\sigma_4 k}{6a^6}}{\frac{\sigma_1}{6} + \frac{\sigma_3 k^2}{6a^4} + \frac{\sigma_4 k}{a^6}} = -\frac{6x_1 - 2x_3 - x_4}{6(x_1 - x_3 + x_4)}. \quad (5.79)$$

Note that the corresponding quantities for dark matter coincide with those of the detailed balance case (expressions (5.36) and (5.38)).

Using the aforementioned auxiliary variables, the Friedmann equations (5.28), (5.29) become:

$$1 = x_1 + x_2 + x_3 + x_4 + x^2 + y^2 \quad (5.80)$$

$$\frac{H'}{H} = -3x^2 - x_2 - 2x_3 - 3x_4. \quad (5.81)$$

Table 5.10: The singular points of a universe governed by Hořava gravity beyond detailed balance (system 5.82)) and their behavior. NH stands for nonhyperbolic (adapted from [151]).

Cr. P	$x_{2c}$	$x_{3c}$	$x_{4c}$	$x_c$	$y_c$	Existence	Stable for
$P_{23}$	0	0	$1 - x_c^2$	$x_c$	0	All $q$	NH
$P_{24,25}$	0	0	0	$\pm 1$	0	All $q$	NH
$P_{26}$	0	0	0	0	0	All $q$	stable
$P_{27,28}$	0	0	0	$\sqrt{\frac{2}{3}}q$	$\pm \sqrt{1 - \frac{2q^2}{3}}$	$q^2 \leq \frac{3}{2}$	unstable
$P_{29}$	1	0	0	0	0	All $q$	unstable
$P_{30,31}$	$1 - \frac{1}{2q^2}$	0	0	$\frac{1}{\sqrt{6}q}$	$\pm \frac{1}{\sqrt{3}q}$	$q \neq 0$	unstable
$P_{32}$	0	1	0	0	0	All $q$	unstable
$P_{33,34}$	0	$1 - \frac{1}{q^2}$	0	$\frac{\sqrt{\frac{3}{2}}}{q}$	$\pm \frac{1}{\sqrt{3}q}$	$q \neq 0$	unstable
$P_{35}$	0	0	$1 - \frac{3}{2q^2}$	$\frac{\sqrt{\frac{3}{2}}}{q}$	0	All $q$	NH

Thus, after using the first of these relations in order to eliminate one variable, the corresponding autonomous system writes:

$$\begin{aligned}
x'_2 &= 2x_2 (3x^2 + x_2 + 2x_3 + 3x_4 - 1), \\
x'_3 &= 2x_3 (3x^2 + x_2 + 2x_3 + 3x_4 - 2), \\
x'_4 &= 2x_4 (3x^2 + x_2 + 2x_3 + 3x_4 - 3), \\
x' &= 3x^3 + (x_2 + 2x_3 + 3x_4 - 3)x + \sqrt{6}qy^2, \\
y' &= \left(3x^2 - \sqrt{6}qx + x_2 + 2x_3 + 3x_4\right)y,
\end{aligned} \tag{5.82}$$

defining a dynamical system in  $\mathbb{R}^5$ . Its singular points and their properties are displayed in table 5.10 and in table 5.11 we present the corresponding observable cosmological quantities.

The curve of nonhyperbolic singular points denoted by  $P_{23}$  is “normally hyperbolic” [368] because they give rise to the eigenvalues  $\{6, 0, 2, 4, 3 - \sqrt{6}qx_c\}$  and the eigenvector associated to the zero eigenvalues is tangent to the set.<sup>1</sup> Thus, examining the sign of the real parts of the non-null eigenvalues, we find that they are always local sources provided  $qx_c > \sqrt{6}/2$ . Amongst all nonhyperbolic singular points,  $P_{26}$ , whose eigenvalues are  $\{-6, -4, -3, -2, 0\}$ , proves to be a stable one (this result was

<sup>1</sup>This curve contains the singular points  $P_{24,25}$  for the choice  $x_c = \pm 1$ , and  $P_{35}$  for the choice  $x_c = \frac{\sqrt{\frac{3}{2}}}{q}$ . In general for a singular point located at  $P_{23}$  we can obtain accurate information about its stability by using the center manifold theory.

Table 5.11: Observable cosmological quantities of a universe governed by Hořava gravity beyond detailed balance.

Cr. P	$w_M$	$\Omega_M$	$\Omega_{DE}$	$w_{DE}$
$P_{23}$	1	$x_c^2$	$1 - x_c^2$	$1/6$
$P_{24,25}$	1	1	0	arbitrary
$P_{26}$	arbitrary	0	1	-1
$P_{27,28}$	$\frac{4q^2}{3} - 1$	1	$\frac{-2q^2 + \sqrt{9-6q^2+3}}{-2q^2 + \sqrt{9-6q^2+6}}$	-1
$P_{29}$	1	1	0	arbitrary
$P_{30,31}$	$-\frac{1}{3}$	$\frac{1}{2q^2}$	$\frac{\sqrt{3}q+1}{3q^2+\sqrt{3}q+1}$	-1
$P_{32}$	arbitrary	0	1	$1/3$
$P_{33,34}$	$\frac{1}{3}$	$\frac{1}{q^2}$	$1 + \frac{3}{(\sqrt{3}-3q)q-1}$	$\frac{2-q(q+\sqrt{3})}{(\sqrt{3}-3q)q+2}$
$P_{35}$	1	$\frac{3}{2q^2}$	$1 - \frac{3}{2q^2}$	$\frac{1}{6}$

proved in [151] using Normal Forms calculations). We will present next the stability analysis of its center manifold.  $P_{27,28}$  are saddle points and their stable manifold can be 4-dimensional provided  $-\frac{\sqrt{2}}{2} < q < \frac{\sqrt{2}}{2}$  because they give rise to the eigenvalues  $\{4q^2, -3 + 2q^2, 2(-3 + 2q^2), 4(-1 + q^2), 2(-1 + 2q^2)\}$ .  $P_{29}$  has eigenvalues  $\{-4, -2, -2, 2, 1\}$ , thus, it admits a 2-dimensional unstable manifold tangent to the  $x_2$ - $y$  plane and its stable manifold is always 3-dimensional. The eigenvalues of the linearization around  $P_{30,31}$  are  $\{-4, -2, 2, -1 - \sqrt{-3 + \frac{2}{q^2}}, -1 + \sqrt{-3 + \frac{2}{q^2}}\}$ , thus, they have a 4-dimensional stable manifold provided  $q^2 > \frac{2}{3}$  or  $-\sqrt{\frac{2}{3}} \leq q \leq -\frac{\sqrt{2}}{2}$  or  $\frac{\sqrt{2}}{2} < q \leq \sqrt{\frac{2}{3}}$ . The point  $P_{32}$  is a saddle point since its eigenvalues are  $\{-4, -2, 2, 2, -1\}$ . Finally,  $P_{33,34}$  has a 3-dimensional stable manifold if  $q^2 > \frac{16}{15}$  or  $-\frac{4}{\sqrt{15}} \leq q < -1$  or  $1 < q \leq \frac{4}{\sqrt{15}}$  because they gives rise to the eigenvalues  $\{4, -2, 2, -\frac{1}{2}\left(1 - \sqrt{-15 + \frac{16}{q^2}}\right), -\frac{1}{2}\left(1 + \sqrt{-15 + \frac{16}{q^2}}\right)\}$ .

#### 5.4.1 Stability Analysis of the *de Sitter* Solution in Hořava-Lifshitz cosmology

In order to analyze the stability of *de Sitter* solution we can use center manifold theorem.

**Proposition 19** *The origin for the system (5.82) is locally asymptotically stable.*

In order to determine the local center manifold of (5.82) at the origin we have to transform the system into a form suitable for the application of the center manifold theorem

(see section 2.2.5.3 for a summary of the techniques involved in the proof).

**Proof.** In order to transform the linear part of the vector field into its Jordan canonical form, we define new variables  $(u, v_1, v_2, v_3, v_4) \equiv \mathbf{x}$ , by the equations

$$u = y, v_1 = x_4, v_2 = x_3, v_3 = x, v_4 = x_2$$

so that

$$\begin{pmatrix} u' \\ v_1' \\ v_2' \\ v_3' \\ v_4' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} u \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} + \begin{pmatrix} f(u, v_1, v_2, v_3, v_4) \\ g_1(u, v_1, v_2, v_3, v_4) \\ g_2(u, v_1, v_2, v_3, v_4) \\ g_3(u, v_1, v_2, v_3, v_4) \\ g_4(u, v_1, v_2, v_3, v_4) \end{pmatrix} \quad (5.83)$$

where

$$\begin{aligned} f(u, v_1, v_2, v_3, v_4) &= u (3v_1 + 2v_2 - \sqrt{6}qv_3 + 4v_4), \\ g_1(u, v_1, v_2, v_3, v_4) &= 2v_1 (3v_3^2 + 3v_1 + 2v_2 + v_4), \\ g_2(u, v_1, v_2, v_3, v_4) &= 2v_2 (3v_3^2 + 3v_1 + 2v_2 + v_4), \\ g_3(u, v_1, v_2, v_3, v_4) &= \sqrt{6}qu^2 + v_3 (3v_3^2 + 3v_1 + 2v_2 + v_4), \text{ and } g_4(x, y_1, y_2, y_3) = \\ &= 2v_4 (3v_3^2 + 3v_1 + 2v_2 + v_4). \end{aligned}$$

The system (5.83) is written in diagonal form

$$\begin{aligned} u' &= Cu + f(u, \mathbf{v}) \\ \mathbf{v}' &= P\mathbf{v} + \mathbf{g}(u, \mathbf{v}), \end{aligned} \quad (5.84)$$

where  $(u, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^4$ ,  $C$  is the zero  $1 \times 1$  matrix,  $P$  is a  $4 \times 4$  matrix with negative eigenvalues and  $f, \mathbf{g}$  vanish at  $\mathbf{0}$  and have vanishing derivatives at  $\mathbf{0}$ . The center manifold theorem 13 asserts that there exists a 1-dimensional invariant local center manifold  $W^c(\mathbf{0})$  of (5.84) tangent to the center subspace (the  $\mathbf{v} = \mathbf{0}$  space) at  $\mathbf{0}$ . Moreover,  $W^c(\mathbf{0})$  can be represented as

$$W^c(\mathbf{0}) = \{(u, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^4 : \mathbf{v} = \mathbf{h}(u), |u| < \delta\}; \quad \mathbf{h}(0) = \mathbf{0}, D\mathbf{h}(0) = \mathbf{0}\}$$

for  $\delta$  sufficiently small (see definition 13). The restriction of (5.84) to the center manifold is (see definition 2.36)

$$u' = f(u, \mathbf{h}(u)). \quad (5.85)$$

According to Theorem 14, if the origin  $x = 0$  of (5.85) is stable (asymptotically stable) (unstable) then the origin of (5.84) is also stable (asymptotically stable) (unstable). Therefore, we have to find the local center manifold, i.e., the problem reduces to the computation of  $\mathbf{h}(u)$ .

Substituting  $\mathbf{v} = \mathbf{h}(u)$  in the second component of (5.84) and using the chain rule,  $\mathbf{v}' = D\mathbf{h}(u)u'$ , one can show that the function  $\mathbf{h}(u)$  that defines the local center manifold satisfies

$$D\mathbf{h}(u)[f(u, \mathbf{h}(u))] - P\mathbf{h}(u) - \mathbf{g}(u, \mathbf{h}(u)) = 0. \quad (5.86)$$

According to Theorem 15, equation (5.86) can be solved approximately by using an approximation of  $\mathbf{h}(u)$  by a Taylor series at  $u = 0$ . Since  $\mathbf{h}(0) = \mathbf{0}$  and  $D\mathbf{h}(0) = \mathbf{0}$ , it is obvious that  $\mathbf{h}(u)$  commences with quadratic terms. We substitute

$$\mathbf{h}(x) =: \begin{bmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \\ h_4(x) \end{bmatrix} = \begin{bmatrix} a_1u^2 + a_2u^3 + a_3u^4 + O(u^5) \\ b_1u^2 + b_2u^3 + b_3u^4 + O(u^5) \\ c_1u^2 + c_2u^3 + c_3u^4 + O(u^5) \\ d_1u^2 + d_2u^3 + d_3u^4 + O(u^5) \end{bmatrix}$$

into (5.86) and set the coefficients of like powers of  $u$  equal to zero to find the unknowns  $a_1, b_1, c_1, d_1, \dots$

We find that the non-zero coefficients are

$$c_1 = \sqrt{\frac{2}{3}}q, \quad c_3 = \frac{4}{3}\sqrt{\frac{2}{3}}q^3,$$

Therefore, (5.85) yields

$$u' = -2q^2u^3 + \left(2q^2 - \frac{8q^4}{3}\right)u^5 + O(u^6). \quad (5.87)$$

Neglecting the error terms, this is a gradient-like equation (i.e.,  $u' = -\nabla U(u)$ ) with potential  $U(u) = \frac{1}{9}q^2(4q^2 - 3)u^6 + \frac{q^2u^4}{2}$  for which the origin is a degenerate minimum. Thus, the origin  $u = 0$  of (5.87) is locally asymptotically stable. Hence, the origin  $\mathbf{u} = \mathbf{0}$  of the full five-dimensional system is asymptotically stable. ■

#### 5.4.2 Cosmological implications: Beyond detailed balance

Let us now discuss about the cosmological behavior of a Hořava-Lifshitz universe, in the case where the detailed balance condition is abandoned. In this case the system admits the unstable singular points  $P_{27,28,29}$  which correspond to dark matter domination, the unstable point  $P_{32}$  corresponding to an unphysical dark-energy dominated universe, and the unstable  $P_{30,31,33,34}$  which have physical  $w_M, w_{DE}$  but dependent on the specific dark-matter form. The system admits also the singular points  $P_{23}, P_{35}$  which are nonhyperbolic with positive non-null eigenvalues, thus unstable, with furthermore unphysical cosmological quantities. Additionally, points  $P_{24,25}$  are also dark-matter dominated, unstable nonhyperbolic ones.

It is interesting to notice that since  $\sigma_3$  has an arbitrary sign,  $P_{33,34}$  could also corre-



spond to an oscillatory universe, for a wide region of the parameters  $\sigma_3$  and  $q$ . However, this oscillatory behavior has a small probability to be the late-time state of the universe because it is not stable (with at least two positive eigenvalues). Additionally, the fact that it depends on  $q$  means that this solution depends on the matter form of the universe.

The scenario at hand admits a final singular point, namely  $P_{26}$ . As we showed in detail in section 5.4 using Center Manifold Theory, it is indeed asymptotically stable and thus it can be a late-time attractor of Hořava-Lifshitz universe beyond detailed balance. Using the definition of the auxiliary variables, we can straightforwardly show that it corresponds to an eternally expanding solution. Additionally, it is characterized by complete dark energy domination, with dark-energy equation-of-state parameter  $-1$  and arbitrary  $w_M$ . Note also that this result is independent of the specific form of the dark-matter content. These features make it a very good candidate for the description of our universe. We mention that according to the initial conditions, this universe on its way towards this late-time attractor can be just an expanding universe with a non-negligible dark matter content, which is in agreement with observations, and this can be verified also by numerical investigation. This fact makes the aforementioned result more concrete.

## 5.5 Conclusions

In this work we performed a detailed phase-space analysis of Hořava-Lifshitz cosmology, with and without the detailed-balance condition. In particular, we examined if a universe governed by Hořava gravity can have late-time solutions compatible with observations.

In the case where the detailed-balance condition is imposed, we find that the universe can reach a bouncing-oscillatory state at late times, in which dark-energy, behaving as a simple cosmological constant, will be dominant. Such solutions were already investigated in the context of Hořava-Lifshitz cosmology [282, 283, 284] as possible ones, but now we see that they can indeed be the late-time attractor for the universe. They arise purely from the novel terms of Hořava-Lifshitz cosmology, and in particular the dark-radiation term proportional to  $a^{-4}$  is responsible for the bounce, while the cosmological constant term is responsible for the turnaround.

In the case where the detailed-balance condition is abandoned, we find that the universe reaches an eternally expanding solution at late times, in which dark-energy, behaving like a cosmological constant, dominates completely. Note that according to the initial conditions, the universe on its way to this late-time attractor can be an expanding one with non-negligible matter content. We mention that this behavior is independent of the specific form of the dark-matter content. Thus, the aforementioned features make this scenario a good candidate for the description of our universe, in consistency with observations. Finally, in this case the universe has also a probability to reach an oscillatory solution at late times, if the initial conditions lie in its basin of attraction (in this case the

eternally expanding solution will not be reached).

Although this analysis indicates that Hořava-Lifshitz cosmology can be compatible with observations, it does not enlighten the discussion about possible conceptual and phenomenological problems and instabilities of Hořava-Lifshitz gravity, nor it can interfere with the questions concerning the validity of its theoretical background, which is the subject of interest of other studies. It just faces the problem from the cosmological point of view, and thus its results can be taken into account only if Hořava gravity passes successfully the aforementioned theoretical tests.

# Chapter 6

## Cardassian Cosmologies

In this chapter we analyze the asymptotic behavior of Cardassian cosmological models filled with a perfect fluid and a scalar field with an exponential potential. Cardassian cosmologies arise from modifications of the Friedmann equation, and among the different proposals within that framework we will choose those of the form  $3H^2 - \rho \propto \rho^n$  with  $n < 1$ . We construct a three dimensional dynamical systems arising from the evolution equations. Using standard dynamical systems techniques we find the fixed points and characterize the solutions they represent. We pay especial attention to the properties inherent to the modifications and compare with the (standard) unmodified scenario. Among other interesting results, we find there are no late-time scaling attractors.

### 6.1 Introduction

Cardassian cosmologies are non-relativistic phenomenological models without a covariant formulation as those based in General Relativity and they have a different justification than quintessence. However, with regard to the observational tests that depend only on the scale or the Hubble factor, in the late-time regime Cardassian models filled with just matter ( $\rho \propto a^{-3}$ ) are indistinguishable from perfect fluid models with a  $p = (\gamma - 1)\rho$  equation of state under the identification  $n \equiv \gamma$ . These perfect fluid models are in turn kinematically equivalent to scalar field (quintessence) models with an exponential potential. In this way, the Cardassian model can make contact with quintessence with regard to observational tests. Interestingly, observational tests seem to favor  $n < 0$ , so that asymptotically one would get a phantom equation of state [166, 167, 168, 177, 185, 188, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450]. However, the equivalence between Cardassian and perfect fluid models is not extensible to the dynamical realm, the evolution of perturbations may differ significantly, and this can lead to discrepancies for instance in observational tests associated with the cosmic microwave background. Nevertheless, as stated in [41], questions of interpretation remain open, be-

cause in the Cardassian model matter alone is responsible for the accelerated behavior, and yet the universe can be flat. The condition for acceleration is  $n < 2/3$ .

Dynamical systems techniques have been using for exploring Cardassian models filled with baryonic matter in [451]. However, in [452] was given a step further by allowing as well for a scalar field component (non-baryonic matter). As well as in [452] in this chapter we investigate for early and late-time tracking (scaling) solutions; i.e., such that both baryonic and non-baryonic matter contributes with non-negligible and proportional fractions to the critical density. Tracking solutions are particularly interesting because their dynamical effects mimics a decaying cosmological constant (see the classical references [194, 197, 409, 410, 453, 454, 455, 456]). Such solutions would be devoid of the fine-tuning problems posed by a cosmological constant precisely because of the independence on the initial conditions.

For this study we consider a self-interacting exponential potential [194, 197, 409, 410, 453, 454, 455, 456, 457, 458, 459] for allowing the reduction of the dimension of the resulting autonomous dynamical systems [460].

For the case  $0 < n < 2/3$  it is possible to make comparisons with [409]. However, since the recent observations favors the case  $n < 0$  [461, 462, 463, 464, 465] we also consider these values in our numerical simulations.

## 6.2 Field equations

The evolution equations for a flat Friedmann-Robertson-Walker (FRW) Cardassian cosmological model filled with a scalar field  $\phi$  with self-interaction potential  $V(\phi) = V_0 \exp(-s\phi)$  and a barotropic perfect fluid with equation of state  $p_\gamma = (\gamma - 1)\rho_\gamma$  are

$$2\dot{H} + \left(\gamma\rho_\gamma + \dot{\phi}^2\right) (1 + n\sigma\rho_{\text{tot}}^{n-1}) = 0 \quad (6.1)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0, \quad (6.2)$$

$$\dot{\rho}_\gamma + 3\gamma H\rho_\gamma = 0, \quad (6.3)$$

where for the total energy density  $\rho_{\text{tot}}$  we have

$$\rho_{\text{tot}} = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_\gamma. \quad (6.4)$$

The evolution equations (6.1-6.3) are in turn subject to the constraint

$$H^2 = \frac{1}{3}\rho_{\text{tot}} (1 + \sigma\rho_{\text{tot}}^{n-1}). \quad (6.5)$$

Here and throughout  $\sigma$ ,  $V_0$  and  $s$  will be free parameters, and we will restrict ourselves to the  $n < 1$  case.

### 6.3 Phase-space analysis

Experience has demonstrated that dynamical systems methods can be used to describe the evolution of cosmological models by means of past and future attractors. In order to cast our set of equations as a dynamical system, it is convenient to normalize the variables, because in the vicinity of an hypothetical initial singularity physical variables would typically diverge, whereas at late times they commonly tend to zero [411]. Due to physical considerations normalization with the Hubble factor is an appropriate choice in cosmology. Besides, all available mathematical evidence suggests that Hubble-normalized variables are bounded into the past (that is, as the initial singularity is approached), and if there is a cosmological constant (or something that mimics it) it seems those variables will also be bounded into the future. Thus, even though the Hubble-normalized state space is unbounded, it is sensible to expect that the evolution equations will admit a past attractor and a future attractor.

Let us introduce the normalized variables

$$w = \frac{\sigma \rho_{\text{tot}}^n}{3H^2}, x = \frac{\dot{\phi}}{\sqrt{6}H}, y = \frac{\sqrt{V}}{\sqrt{3}H}, z = \frac{\sqrt{\rho_\gamma}}{\sqrt{3}H} \quad (6.6)$$

This coordinates will allow us analyzing the solutions of (6.1-6.2), and the cosmological models associated with them. In addition, the variables will be related among them through

$$w + x^2 + y^2 + z^2 = 1. \quad (6.7)$$

The constraint (6.7) lets us “forget” about the evolution of one of the coordinates. Here we will choose the discarded coordinate to be  $w$ . Using the variables (6.6), equation (6.7), and the conservation equations (6.2) and (6.3) we get the equations

$$x' = \frac{3nx(z^2(\gamma - 2) - 2y^2)}{2(x^2 + y^2 + z^2)} + \frac{1}{2} \left( \sqrt{6}sy^2 - 3(n - 1)x(2x^2 + z^2\gamma - 2) \right), \quad (6.8)$$

$$y' = \frac{3ny(2x^2 + z^2\gamma)}{2(x^2 + y^2 + z^2)} - \frac{1}{2}y \left( \sqrt{6}sx + 3(n - 1)(2x^2 + z^2\gamma) \right), \quad (6.9)$$

$$z' = \frac{3nz(2x^2 + z^2\gamma)}{2(x^2 + y^2 + z^2)} - \frac{3}{2}z \left( 2(n - 1)x^2 + ((n - 1)z^2 + 1)\gamma \right). \quad (6.10)$$

Observe that the equations are invariant under the variable changes  $y \rightarrow -y$ , and  $z \rightarrow -z$ , but not under  $x \rightarrow -x$ , and so in our numerical examples we will concentrate on the region  $\{x^2 + y^2 + z^2 \leq 1, -1 \leq x \leq 1, y \geq 0, z \geq 0\}$ . This is equivalent to saying we are just considering expanding universes ( $H > 0$ ). However, an analytic description of all singular points is presented in the lines below and in table 6.1. We will also set restrictions  $0 < \gamma < 2$  and  $s^2 < 6$  so that neither the barotropic fluid nor the scalar field

have supraluminal sound speeds and the fluid satisfies the weak energy condition. We will also assume  $n < 1$ , as this is the case of interest.

In table 6.1 are displayed the coordinates, existence conditions and stability conditions of the critical points of the dynamical system (6.8)-(6.10). Although the location of the critical points of this dynamical system does not depend on  $n$ <sup>1</sup>, the same is not true for their dynamical character. In table 6.1 are displayed the eigenvalues of the Jacobian matrix evaluated at each critical points. Due to the symmetries of the equations (6.8)-(6.10), the system can be simplified by performing the coordinate transformation to spherical coordinates  $\{r, \theta, \varphi\}$ , i.e.,

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta.$$

Our region of interest is

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 < \varphi < \pi.$$

Equations (6.8)-(6.10) becomes in spherical coordinates to

$$r' = -\frac{3}{4}(n-1)r(r^2-1)(2\cos(2\varphi)\sin^2\theta + \gamma + (\gamma-1)\cos(2\theta) + 1), \quad (6.11)$$

$$\theta' = -\frac{3}{2}\cos\theta(-\gamma + \cos(2\varphi) + 1)\sin\theta, \quad (6.12)$$

$$\varphi' = \frac{1}{2}(6\cos(\varphi) - \sqrt{6}rs\sin\theta)\sin(\varphi). \quad (6.13)$$

The origin of coordinates ( $P_5$ ) corresponds to  $r = 0$ . Its dynamical character cannot be determined using the linearization. In fact, for  $r = 0, \theta = 0, \varphi = 0$ , the eigenvalues of the linearization of (6.11)-(6.13) are  $3, \frac{3(\gamma-2)}{2}, \frac{3}{2}(n-1)\gamma$ , whereas for  $r = 0, \theta = \pi/2, \varphi = \pi/2$ , they are  $-3, 0, -\frac{3\gamma}{2}$ . Since both are different representations for the origin, its dynamical character cannot be anticipated from the linearization. This is due to the fact that the dynamical system is not of class  $C^1$  at the origin. Based in numerical investigation (see figures 6.1, 6.2) we can support the claim that the origin is a local sink for (6.8)-(6.10).

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<sup>1</sup>This is confirmed by numerical tests.

Table 6.1: Location and existence conditions, eigenvalues and dynamical character of the critical points of the dynamical system (6.8)-(6.10). We assume  $n < 1$ ,  $0 < \gamma < 2$  and  $s^2 < 6$ . We use the notation  $\beta_{\pm} = -\frac{3}{4} \left( (2 - \gamma) \pm \sqrt{(2 - \gamma)(24\gamma^2/s^2 + (2 - 9\gamma))} \right)$ .

Name	Coordinates: $x, y, z$	Existence	Eigenvalues	Dynamical Character
$P_1$	$0, 0, 1$	All $\gamma$ and $s$	$\frac{3\gamma}{2}, 3(1 - n)\gamma, -\frac{3(2-\gamma)}{2}$	saddle.
$P_2^{\pm}$	$\pm 1, 0, 0$	All $\gamma$ and $s$	$6(1 - n), 3 \mp \sqrt{\frac{3}{2}s}, \frac{3(2-\gamma)}{2}$	source for $\pm s < \sqrt{6}$ ; saddle otherwise.
$P_3$	$\frac{s}{\sqrt{6}}, \sqrt{1 - \frac{s^2}{6}}, 0$	$s^2 < 6$ saddle otherwise.	$-\frac{(6-s^2)}{2}, \frac{(s^2-3\gamma)}{2}, (1 - n)s^2$	non-hyp. for $s^2 = 3\gamma$ or $s = 0$ ;
$P_4$	$\sqrt{\frac{3}{2}}\frac{\gamma}{s}, \sqrt{\frac{3(2-\gamma)\gamma}{2s^2}}, \sqrt{1 - \frac{3\gamma}{s^2}}$	$s^2 > 3\gamma$ saddle otherwise.	$3(1 - n)\gamma, \beta_+, \beta_-$	non-hyp. for $\gamma = 0, 2$ ;
$P_5$	$0, 0, 0$	All $\gamma$ and $s$	undefined	local sink (see text).

**Proposition 20** *The origin of coordinates of the dynamical system (6.8)-(6.10) is:*

1. *stable for initial solutions outside the invariant set  $y = 0$ ;*
2. *asymptotically stable for solutions in the invariant set  $y = 0$ .*

**Proof.**

In the limit  $r \rightarrow 0$ , equations (6.12), (6.13) reduce to

$$\theta' = \frac{3}{4}(\gamma - \cos(2\varphi) - 1) \sin(2\theta), \quad (6.14)$$

$$\varphi' = -\frac{3}{2} \sin(2\varphi). \quad (6.15)$$

The approximated system (6.14)-(6.15) is completely integrable with solution

$$\begin{aligned} \theta(\tau) &= \tan^{-1} \left( e^{\frac{3}{2}(\gamma-2)\tau+2c_2} \sqrt{1 + e^{6\tau+4c_1}} \right), \\ \varphi(\tau) &= \tan^{-1} \left( e^{2c_1-3\tau} \right), \end{aligned}$$

where  $c_1$  and  $c_2$  are integration constants.

Taylor expanding (6.11) around  $r = 0$  we obtain

$$r' = \frac{3}{4}(n-1) \left( 2 \cos(2\varphi) \sin^2 \theta + \gamma + (\gamma-1) \cos(2\theta) + 1 \right) r + O(r^3) \quad (6.16)$$

By substituting the previous first order solutions for  $\theta(\tau), \varphi(\tau)$  in (6.16) and integrating the resulting equation we obtain

$$\rho(\tau) = e^{\frac{3}{2}(n-1)\gamma\tau} \left( 1 + e^{3(\gamma-2)\tau+4c_2} + e^{3\gamma\tau+4(c_1+c_2)} \right)^{\frac{1-n}{2}} c_3.$$

Passing to Cartesian coordinates we have that

$$\begin{aligned} x &= a_2 e^{\frac{3}{2}(n\gamma-2)\tau} \left( e^{3\gamma\tau} a_1^2 + a_2^2 e^{3(\gamma-2)\tau} + 1 \right)^{-n/2} c_3, \\ y &= a_1 e^{\frac{3n\gamma\tau}{2}} \left( e^{3\gamma\tau} a_1^2 + a_2^2 e^{3(\gamma-2)\tau} + 1 \right)^{-n/2} c_3, \\ z &= e^{\frac{3}{2}(n-1)\gamma\tau} \left( e^{3\gamma\tau} a_1^2 + a_2^2 e^{3(\gamma-2)\tau} + 1 \right)^{-n/2} c_3, \end{aligned}$$

where we have introduced the reescaling

$$c_1 \rightarrow \frac{\log(|a_1|)}{2} - \frac{\log(|a_2|)}{2}, \quad c_2 \rightarrow \frac{\log(|a_2|)}{2}.$$

Assuming when  $n < 1$ , and taking the limit as  $\tau \rightarrow +\infty$  in the above expressions, we obtain that  $x$  and  $z$  approaches to zero at an exponential rate, whereas  $y \rightarrow \bar{y} \equiv y_0^{1-n} (x_0^2 + y_0^2 + z_0^2)^{n/2}$ , where  $x(0) = x_0, y(0) = y_0, z(0) = z_0$ . Let us assume that  $n < 1$  and let be  $\epsilon > 0$  an arbitrary number. Then there exists a  $\delta > 0$ , such that  $\delta < \epsilon$ .



Let us consider the solution with initial value  $x(0) = x_0, y(0) = y_0, z(0) = z_0$ , with  $x_0^2 + y_0^2 + z_0^2 < \delta^2$ . Since  $y \rightarrow \bar{y}$ , satisfying  $|\bar{y}| < \delta$ , then the solution,  $\mathbf{x}(\tau, \mathbf{x}_0)$  passing through  $\mathbf{x}_0 = (x_0, y_0, z_0)$  at  $\tau = 0$ , satisfies  $\|\mathbf{x}(\tau, \mathbf{x}_0)\| < \epsilon$ , for  $\tau$  arbitrarily large. In this way we prove the stability of  $P_5$ . For solutions passing through  $\mathbf{x}_0 = (x_0, y_0, z_0)$  at  $\tau = 0$ , with  $y_0 = 0$ ,  $P_5$  is asymptotically stable. ■

The result of proposition 20 is illustrated numerically in the figures 6.1, 6.2.

## 6.4 Basic observables

In this section we evaluate the deceleration parameter

$$q \equiv -\frac{a\ddot{a}}{\dot{a}^2},$$

and the effective EoS parameter

$$w_{\text{eff}} \equiv \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi) + (\gamma - 1)\rho_\gamma}{\rho_{\text{tot}}},$$

at the singular points.

Table 6.2: Deceleration parameter,  $q$ , and effective EoS,  $w_{\text{eff}}$  at the critical points of the dynamical system (6.8)-(6.10).

Name	$q, w_{\text{eff}}$	Type of solution
$P_1$	$\frac{1}{2}(3\gamma - 2), \gamma - 1$	accelerating for $\gamma < \frac{2}{3}$ ; matter-dominated.
$P_2^\pm$	$2, 1$	decelerating; kinetic-dominated.
$P_3$	$\frac{1}{2}(s^2 - 2), \frac{1}{3}(s^2 - 3)$	accelerating for $s^2 < 2$ ; quintessence.
$P_4$	$\frac{3\gamma}{2} - 1, \gamma - 1$	accelerating for $\gamma < \frac{2}{3}$ ; matter-scalar scaling.
$P_5$	undefined	Cardassian corrections

Using the variables (6.6), and equation (6.7) we obtain

$$q = \frac{3n(2x^2 + z^2\gamma)}{2(x^2 + y^2 + z^2)} + \frac{1}{2}(-6(n-1)x^2 - 3(n-1)z^2\gamma - 2)$$

and

$$w_{\text{eff}} = \frac{x^2 - y^2 + z^2(\gamma - 1)}{x^2 + y^2 + z^2}.$$

In table 6.2 are displayed the deceleration parameter,  $q$ , and the effective EoS,  $w_{\text{eff}}$  at the critical points of the dynamical system (6.8)-(6.10).

## 6.5 Physical interpretation

In what follows, and in order to complete the information provided in the tables, we will characterize the cosmological models represented by the singular point living in the above mentioned phase-space.

The first point, called  $P_1$  (equivalent to  $W_+$  in the notation of [452]), represents a solution completely dominated by the fluid. The unstable character of these solutions agrees with what one might have anticipated, as they are only expected to be relevant at early times.

The second point,  $P_2^\pm$  (equivalent to  $X_\pm$  in the notation of [452]), represents a solution completely dominated by the scalar field, more specifically by its kinetic energy. As discussed in table 6.1 they are either saddles or sources.

The third point, called  $P_3$  (equivalent to  $XY_+$  in the notation of [452]), represents a scalar field dominated solution, which is inflationary if  $s^2 < 2$  [149]. They are unstable saddle in the asymptotic future.

The fourth point, called  $P_4$  (equivalent to  $XY_+W$  in the notation of [452]), depicts a tracking solution, neither the fluid nor the scalar field dominate completely [417, 409].

The fifth point, called  $P_5$  (equivalent to O in the notation of [452]), represents a regime where the Cardassian corrections dominates. As a difference with the analysis in [452] where this point is non-hyperbolic, in the present study it is possible to completely characterize this point using spherical coordinates. The critical point  $P_5$  is stable for initial solutions outside the invariant set  $y = 0$ , whereas, for solutions in the invariant set  $y = 0$ , it is asymptotically stable. Thus, for a massless scalar field,  $P_5$  is the local sink. We have presented several numerical experiments to show this feature.

## 6.6 Cosmological consequences

The cosmological consequences of this analysis are simple but important. As compared to the situation in standard cosmology, for the description of Cardassian models we find that the first complication stems from the necessity of introducing an additional variable, which we call  $w$ .

Our numerical analysis tell us that the past attractors correspond to  $x^2 + y^2 + z^2 \equiv 1$ , and because of the constraint the latter enforces  $w = 0$  which in turn implies the recovery of the usual form of the Friedmann equation. In the case of models expanding from an initial singularity will and for ( $n < 1$ ) we then conclude that the past attractors corresponds more specifically to  $w = 0$ , and from the definition of  $w$  we see that those are solutions with an initial singularity. Summarizing, from the perspective of dynamical systems Cardassian models with a fluid and a scalar field with an exponential potential

will preferably have a big bang.

More specifically, the early-time attractor is a solution completely dominated by the kinetic energy of the scalar field and satisfying  $\rho \propto a^{-6}$ , and its evolution is indistinguishable from that of perfect fluid models with a  $p = (\gamma - 1)\rho$  equation of state under the identification  $n \equiv \gamma/2$ , and the condition for inflation is simply  $n < 1/3$ .

Interestingly, there are no tracking late-time attractors neither de Sitter attractors. The past attractor for a massless scalar field is given by a point where Cardassian corrections dominates. This is an important difference with respect to the behavior in standard (non-Cardassian) models. If the potential of the scalar field is important in the dynamics, then the orbits near the origin do not approach asymptotically the solution dominated by Cardassian corrections. This fact is due to the non-differentiability of the system at the origin.

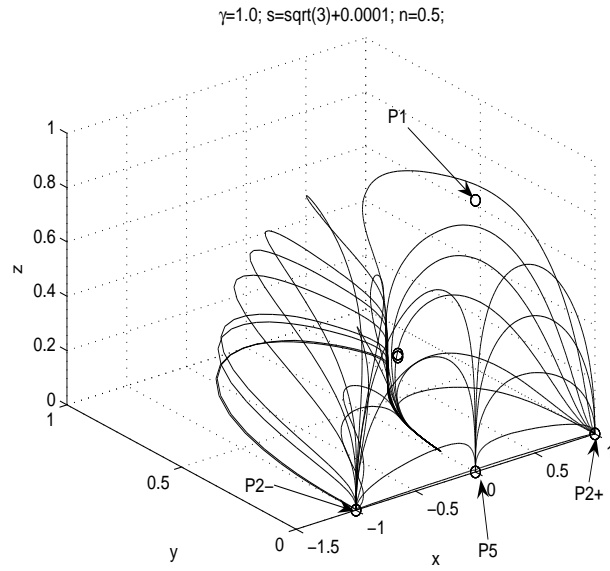
## 6.7 Conclusions

Cardassian models have been proposed as yet one more possible explanation for late-time acceleration. The main interest of the proposal is it involves only matter and radiation and does not invoke either vacuum energy or a cosmological constant. The idea consists in introducing a modification to the Friedmann equation, so that the effects of the modification become important at low redshift.

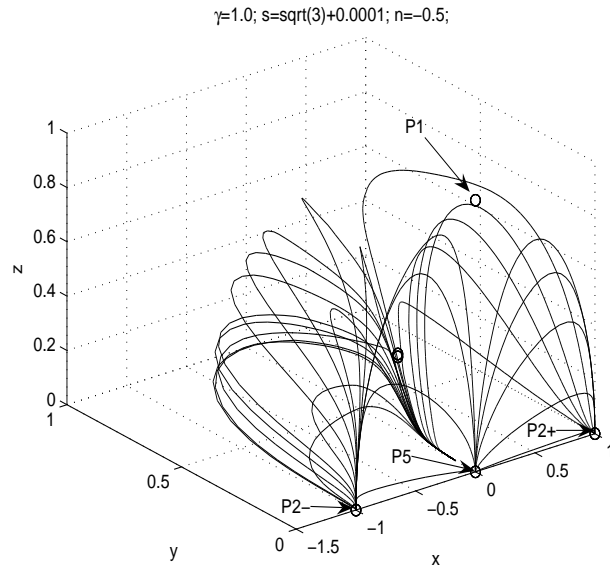
We have concentrated here on modifications of the form  $3H^2 - \rho \propto \rho^n$  with  $n < 1$ , and we have studied its asymptotic behavior assuming  $\rho$  is made up of two contributions: the energy density of a perfect fluid with a  $p = (\gamma - 1)\rho$  equation of state and a self-interacting scalar field with an exponential potential.

Our analysis falls mainly on the analytical side, but we have also carried out some numerical investigations. We constructed a dynamical system arising from the evolution equations.

Our analysis allows us to say that for  $n < 0$ , the late-time solution attractor is a solution completely dominated by the Cardassian corrections which is accelerating for  $n < \frac{2}{3\gamma}$ , and that there are not tracking late-time attractor.

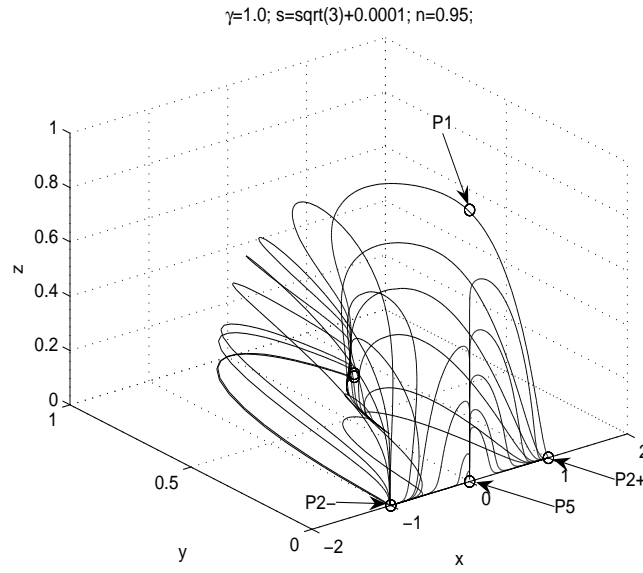


(a)

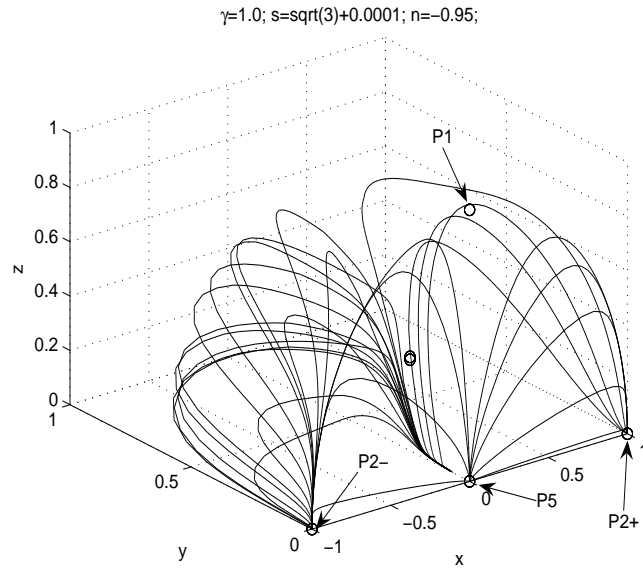


(b)

Figure 6.1: Phase space of Cardassian models for the choices: (a)  $\gamma = 1.0, s = \sqrt{3} + 0.0001, n = 0.5$ . and (b)  $\gamma = 1.0, s = \sqrt{3} + 0.0001, n = -0.5$ .



(a)



(b)

Figure 6.2: Phase space of Cardassian models for the choices: (a)  $\gamma = 1.0, s = \sqrt{3} + 0.0001, n = 0.95$ . and (b)  $\gamma = 1.0, s = \sqrt{3} + 0.0001, n = -0.95$ .



# Conclusions

Dynamical systems provide one of the best ways to study the stability of cosmological models, and crucially, the fine-tuning of initial conditions required to match with observations. The main purpose of the analysis of the flow of a dynamical system is to describe the global dynamics (phase portrait) and the effect of small perturbations of the initial conditions on the dynamics (stability). After an exhaustive bibliographical review on the more recent results on the subject of dynamical studies, in this book we have provided an overview of the applications of these methods to Cosmology. We have used several of the well-known mathematical tools for the stability analysis such that the calculation of Normal forms and the application of Center Manifold Theory for the investigation of non-hyperbolic singular points for which the classical linearization techniques fails to be applied.

The standard procedure to analyze the properties of the flow of a cosmological dynamical system determined by an ordinary differential equation (and some algebraic restrictions) has several steps. To determine whether the state space is compact, since this would imply the existence of both past and future attractors. A nice tool for obtaining a compact phase space is through Poincaré Projection. Identify the lower-dimensional invariant sets, which contain the orbits of more special classes of models with additional symmetries. In order to do that, the devising of monotonic functions is very helpful. Find all the singular points and analyze their local stability. Where possible identify the stable and unstable manifolds of the singular points, which may coincide with some of the invariant sets previously identified. Find Dulac's functions or monotone functions in various invariant sets where possible. Investigate any bifurcations that occur as the equation of state parameter (or any other parameters) varies. The bifurcations are associated with changes in the local stability of the singular points. Having all the information in the previous points one can hope to formulate precise conjectures about the asymptotic evolution, by identifying the past and the future attractors. The past attractor will describe the evolution of a typical universe near the initial singularity while the future attractor will play the same role at late times. The monotone functions, in conjunction with theorems of dynamical systems theory, may enable some of the conjectures to be proved. Knowing the stable and unstable manifolds of the singular points it is possible to construct all possible heteroclinic sequences that join the past attractor, thereby gaining insight into the intermediate evolu-

tion of cosmological models. Having at hand all the available dynamical information of a cosmological model, and using the observational available evidence, one crucially can establish the correspondence between stability conditions, observational evidence and the cosmological structure that we observe.

In this book we have presented some progresses in the theoretical and/or phenomenological modelling of the Universe on the basis of an increasing number of observational data that inform to us into how it is the Universal kinematics on great scales, and on the other hand, in the deepening in the understanding of the fundamental theory that it describes the gravitational interaction.

In order to give a phase space description of several cosmological models we have combined topological, analytical and numerical techniques for obtaining all possible asymptotic behaviors for coupled quintessence dark energy models including or not radiation, based on Scalar-Tensor theories. We have considered mass-varying dark matter-particles in the framework of phantom cosmologies for analyzing the viability of them in order to solve the coincidence problem (why the energy densities of dark matter and of dark energy are comparable in order the magnitude today?) obtaining that this problem cannot be solved or even alleviated for our cases of study. We have obtained all possible asymptotic behaviors for Hořava-Lifshitz cosmologies with and without detailed-balance and finally, we have obtained all possible asymptotic behaviors for the so-called Cardassian cosmologies.

As coupled quintessence models we have considered scalar fields with arbitrary (positive) potentials and arbitrary coupling functions. Then, we have straightforwardly introduced mild assumptions under such functions (differentiable class, number of singular points, asymptotes, etc.) in order to clarify the structure of the phase space of the dynamical system. We have obtained several analytical results. Also, we have presented several numerical evidences that confirm some of these results. We have proved that if the potential is nonnegative and has a local zero minimum at the origin; its derivative and the potential are simultaneously bounded; and provided that the coupling function is of exponential order, then the energy density of the background as well as the kinetic term tend to zero when the time goes forward, meaning the stability of the de Sitter solution. We have devised a monotonic function for the flow of the dynamical system which allow for the identification of some invariant sets. We have provided approximated center manifolds for the vector field around the inflection points and the strict degenerate local minimum of the potential. It is proved that the scalar field typically diverges into the past, generalizing previous results. By assuming some regularity conditions on the potential and on the coupling function in that regime, we have obtained radiation-dominated cosmological solutions; power-law scalar-field dominated inflationary cosmological solutions; matter-kinetic-potential scaling solutions and radiation-kinetic-potential scaling solutions. Scaling attractors are relevant to give an answer to the Coincidence Problem. Also we have



investigated, for the general model including radiation, the important examples of higher order gravity theories  $F(R) = R + \alpha R^2$  (quadratic gravity) and  $F(R) = R^n$ .

In the literature it is reported that non-minimally coupled quintessence models are well suited to give an affirmative answer to the Coincidence problem; thus the natural question would be that if it is possible for phantom cosmological models to alleviate the coincidence problem. In this book, we have investigated phantom cosmological models with dark-matter particles. The dark matter particles acquire masses due to the interaction with the dark-energy sector. We have performed a detailed phase-space analysis of various models, with either exponentially or power-law dependent dark-matter particle mass, in exponential or power-law scalar field potentials. In all the examined cases, solutions having Energy densities of the same orders that might solve the coincidence problem are not relevant attractors at late times. Thus, the coincidence problem cannot be solved or even alleviated in varying-mass dark matter particles models in the framework of phantom cosmology, in a radical contrast with the corresponding quintessence models. Therefore, phantom cosmology with varying-mass dark matter particles cannot easily act as a successful candidate to describe dark energy.

For the case of Hořava gravity we have obtained late-time solutions compatible with observations. In the case where the detailed-balance condition is imposed, we find that the universe can reach a bouncing-oscillatory state at late times, in which dark-energy, behaving as a simple cosmological constant, will be dominant. Such solutions were already investigated in the context of Hořava-Lifshitz cosmology as possible ones, but now we see that they can indeed be the late-time attractor for the universe. They arise purely from the novel terms of Hořava-Lifshitz cosmology, and in particular the dark-radiation term proportional to  $a^4$  is responsible for the bounce, while the cosmological constant term is responsible for the turnaround. In the case where the detailed-balance condition is abandoned, we find that the universe reaches an eternally expanding solution at late times, in which dark-energy, behaving like a cosmological constant, dominates completely. Note that according to the initial conditions, the universe on its way to this late-time attractor can be an expanding one with noneligible matter content. We mention that this behavior is independent of the specific form of the dark-matter content. Thus, the aforementioned features make this scenario a good candidate for the description of our universe, in consistency with observations. Finally, in this case the universe has also a probability to reach an oscillatory solution at late time; if the initial conditions lie in its basin of attraction (in this case the eternally expanding solution will not be reached). Although this analysis indicates that Hořava-Lifshitz cosmology can be compatible with observations, it does not enlighten the discussion about possible conceptual and phenomenological problems and instabilities of Hořava-Lifshitz gravity, nor can it interfere with the questions concerning the validity of its theoretical background, which is the subject of interest of other studies. It just faces the problem from the cosmological point of view, and thus its results can

been taken into account only if Hořava gravity passes successfully the aforementioned theoretical tests.

Finally we have investigated Cardassian models. These have been proposed as yet one more possible explanation for late-time acceleration. The main interest of the proposal is it involves only matter and radiation and does not invoke either vacuum energy or a cosmological constant. The idea consists in introducing a modification to the Friedmann equation, so that the effects of the modification become important at low redshift. Our analysis allows us to say that for  $n < 0$ ; the late-time solution attractor is a solution completely dominated by the Cardassian corrections which can be acceleration; and that there are not tracking late-time attractor.

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